Nonparametric Reliability-based Design Optimization

Using Sign Test on Limited Discrete Information

W. Lim, J. Jang, S. Park, E. Amalnerkar and *T.H. Lee

Department of Automotive Engineering, Hanyang University, 222, Wangsimni-ro, Seongdong-gu, Seoul 133-791, Korea.

*Corresponding author: thlee@hanyang.ac.kr

Abstract

Many methods for reliability analysis and reliability-based design optimization have been developed since the last decades. However, for most of these methods predicting the reliability, stochastic information of the variables has to be assumed as parameters like the mean (location parameter) and variance (scale parameter). This assumption cannot guarantee the accuracy of the reliability when information is limited. In this paper, we propose a nonparametric RBDO using sign test that does not consider the parameter but requires limited discrete information, like only sample data. We define the uncertainty of the reliability as the decision error of nonparametric hypothesis test due to limited information. We examine the tendency of the solution with respect to the reliability of a system and the uncertainty of the reliability through an example.

Keywords: Reliability analysis, Reliability-based design optimization, Uncertainty of reliability, Reliability error, Sign test, Limited discrete information

Introduction

In deterministic design optimization, since uncertainty of input variables does not considered, the reliability of a system cannot be evaluated. Thus, a safety factor has been employed to guarantee the reliability. To consider the reliability of a system, stochastic design optimization such as reliability-based design optimization (RBDO) has been developed. RBDO can provide an optimum point satisfying target the reliability of a system by using stochastic information of input variables.

There are many methods for reliability analysis such as the first and second order reliability method (Cornell, 1969; Breitung, 1984), moment-based method (Lee and Kwak, 2006; Rahman and Xu, 2004) and its implementation to RBDO (Shetty et al., 1998; Shan and Wang, 2007). Generally, in reliability analysis, uncertainty of a system is caused by uncertainty of input variables. In these methods, to treat the uncertainty, distribution of input variables has to be assumed as parameter such as statistical moments. However, in practical problems, information of input variables is often limited and discrete. Thus, the reliability of a system also has uncertainty because of the lack of information. Therefore, to treat the limited discrete information, we should consider the reliability of a system as well as the uncertainty of the reliability.

Recently, RBDO with confidence level under input model uncertainty is suggested (Noh et al., 2011). They assume that the uncertainty of the reliability is due to uncertainty of parameters of input variables and considered as a confidence level of the parameters. However, the method cannot quantify the uncertainty of the reliability but provides qualitative trends of the uncertainty of the reliability.

In this paper, we propose a new approach for nonparametric RBDO (NRBDO) using sign test. We assume that the uncertainty of the reliability is caused by the limited discrete information. A reliability analysis method, for instance, Akaike information criterion (AIC)-based reliability analysis method (Lim and Lee, 2012) is adopted to estimate the reliability of a system for discrete information. With the estimated reliability, we estimate the uncertainty of the reliability by using nonparametric sign test for limited discrete information. The proposed method can quantify the reliability of a system as well as the uncertainty of the reliability. For convenience, we name the reliability of a system and the uncertainty of the reliability as the first and second reliabilities, respectively. Using an example of RBDO with limited and discrete information of input variables, we evaluate the first and second target reliabilities.

We introduce AIC and sign test, then we formulate NRBDO in the second section. In the third section, we illustrate and compare deterministic design optimization (DDO), RBDO and NRBDO for a mathematical example. In the last section conclusions are summarized. When information of input variables is limited and discrete, the proposed method can obtain optimum point considering the reliability of a system as well as the uncertainty of the reliability.

Nonparametric Reliability-based Design Optimization

In this section, we introduce AIC and sign test. AIC is adopted to estimate the reliability using discrete information. Then, considering limited discrete information, sign test is used to decide whether the second reliability namely the uncertainty of the reliability is acceptable or not. Combining these two concepts, we achieve an optimum result satisfying the first as well as the second reliabilities for limited discrete information.

Reliability Analysis Using Akaike Information Criterion

Akaike information criterion was introduced in 1973 by Akaike (Akaike, 1973) and has been developed and implemented on various fields of science such as statistics, ecology, engineering and reliability analysis (Lim and Lee, 2012; Hurvich et al., 1998; Pan, 2001; Spendelow et al., 1995; Al-Rubaie et al., 2007; Go et al., 2011).

AIC is a method that selects the best estimated distribution from candidate distributions provided by a user. The AIC is defined as follows (Sakamoto et al., 1986):

$$\varphi = -2\left(f_{ml} - n_{free}\right) \tag{1}$$

where f_{ml} is maximum log likelihood of a candidate distribution, and n_{free} stands for the number of parameters of a candidate distribution.

In this paper, we use six types of distribution: normal distribution, log-normal distribution, Gamma distribution, Weibull distribution, exponential distribution and generalized extreme value distribution. When the likelihood of a candidate distribution is approaching maximum, the probability of estimation becomes the highest. Hence, we choose the best estimated distribution with the smallest φ. Then we estimate the first reliability by integrating its probability density function.

Second Reliability Analysis Using a Sign Test

The sign test is one of the oldest nonparametric hypothesis tests. In order to make a decision, whether to reject the null hypothesis or not, a binomial distribution is used to calculate the rejection or critical region (Conover, 1980).

Let $P(+)$ and $P(-)$ be the ratio of success and failure, respectively. Then the null and alternative hypothesis of sign test is given as follows:

$$H_0: P(+) \geq P(-) \tag{2}$$

$$H_a: P(-) < P(+)$$

As the number of successes increases, i.e., $P(+)$ increases, decreasing the chances of rejection of the null hypothesis. When we undertake a hypothesis test, a decision error always exists. In this case, there is a type II error since we do not reject null hypothesis. We decide a hypothesis using critical region calculated by integrating the binomial distribution. The probability mass function of binomial distribution is of form as follows:

$$y(x|n_t, p) = \binom{n_t}{x} p^x (1-p)^{n_t - x} \tag{3}$$

where n_t is the number of trials, p stands for success probability, and x is the number of successes which is a non-negative integer.
The critical region can be calculated by a summation of Eq. (3). The summation is of form as follows:

$$R_2 = \sum_{i=0}^{n_s} y(i|n_t, p) \text{ for non-negative integer } i \tag{4}$$

where n_s is the number of success and R_2 stands for the second reliability. Also we can treat Eq. (3) on non-negative real number by using Gamma function. Then Eqs. (3) and (4) can be rewritten as follows:

$$y(x|n_t, p) = \frac{\Gamma(n_t+1)}{\Gamma(n_t-x+1)\Gamma(x+1)} p^x (1-p)^{n_t-x} \tag{5}$$

$$R_2 = \int_0^{n_s} y(x|n_t, p)dx \text{ for non-negative real number } x \tag{6}$$

We can estimate the first reliability using Akaike information criterion, and then estimate the number of success by Eq. (7).

$$n_s = R_1 n_t \tag{7}$$

where R_1 is the first reliability, the reliability of a system. Then, the second reliability using Eq. (6) can be estimated and a decision not to reject the null hypothesis if the second reliability is greater than the second target reliability can then be made.

Mathematical Example

We choose an example for optimization (Noh et al., 2011; Youn and Choi, 2004; Youn and Wang, 2008; Lee et al., 2013) consisting of two design variables and three constraint functions. The optimization formulation of the example is of form in Eq. (8)

Firstly, we find the optimum point using DDO. Secondly, we find the optimum point using RBDO with respect to target reliability. Finally, we find the optimum point using NRBDO with respect to the reliability of a system and the uncertainty of the reliability. In this section, each of the results is illustrated.

$$\begin{aligned}
&\text{Find } x_1, x_2 \\
&\text{to minimize } f(\boldsymbol{x}) = -x_1 + x_2 \\
&\text{subject to } g_1(\boldsymbol{x}) = 1 - \frac{x_1^2 x_2}{20} \\
&\qquad g_2(\boldsymbol{x}) = 1 - \frac{(x_1+x_2-5)^2}{30} + \frac{(x_1-x_2-12)^2}{120} \\
&\qquad g_3(\boldsymbol{x}) = 1 - \frac{80}{(x_1^2+8x_2+5)}
\end{aligned} \tag{8}$$

Deterministic Design Optimization

DDO problem is formulated in Eq. (8). It is easy to obtain the optimum point. We use 'fmincon' in MATLAB to solve the optimization problem. We can obtain minimum value of -5.9955 at $\boldsymbol{x} = (7.7883, 1.7928)$. The active constraint functions are $g_2(\boldsymbol{x})$ and $g_3(\boldsymbol{x})$ at the optimum point. However, DDO cannot consider the uncertainty of the input variables. Therefore, the optimum point can exist in infeasible region if any uncertainty of the input variables occurs.

Reliability-based Design Optimization

RBDO formulation is of form as follows:

$$\begin{aligned}
&\text{Find } x_1, x_2 \\
&\text{to minimize } f = -x_1 + x_2 \\
&\text{subject to } \Pr\left[G_j(\boldsymbol{X}) \le 0\right] \ge R_1^{\text{target}}, \text{j} = 1,2,3 \\
&\qquad G_1(\boldsymbol{X}) = 1 - \frac{X_1^2 X_2}{20} \\
&\qquad G_2(\boldsymbol{X}) = 1 - \frac{(X_1+X_2-5)^2}{30} + \frac{(X_1-X_2-12)^2}{120} \\
&\qquad G_3(\boldsymbol{X}) = 1 - \frac{80}{(X_1^2+8X_2+5)}
\end{aligned} \tag{9}$$

Generally, in deterministic design optimization, the uncertainty of the input variables cannot be considered. Therefore, to treat the uncertainty, we assume distribution of the input variables as the normal distribution, and then obtain 10^6 random samples from the distribution. The assumed distribution is as follows:

$$X_i \sim N(x_i, \sigma^2), i = 1,2, \sigma = 0.3 \tag{10}$$

We use Monte Carlo simulation to estimate the reliability of a system using 10^6 samples. A genetic algorithm is used as the optimization algorithm.

When the first target reliability is 0.5, the result is similar to that of DDO. Because we assume the distribution of the input variables as the normal distribution, the probability of success is almost 1/2. However, if the first target reliability changes, the optimum point obviously changes. The results of optimum point and its objective value with respect to the first target reliability are shown in Table 1. If the design requires higher reliability, then the objective function should have a higher value.

Nonparametric Reliability-based Design Optimization

Using hypothesis test, NRBDO formulation can be written as follows:

$$
\begin{aligned}
&\text{Find } x_1, x_2 \qquad (11)\\
&\text{to minimize } f = -x_1 + x_2\\
&\text{subject to do not reject } H_0\text{: } \Pr\left[G_j(\boldsymbol{X}) \le 0\right] \ge R_1^{\text{target}} \text{ with } R_2^{\text{target}}, \mathrm{j} = 1{,}2{,}3\\
&\qquad R_2^{\text{target}} = 1 - \text{type II error}\\
&\qquad G_1(\boldsymbol{X}) = 1 - \frac{X_1^2 X_2}{20}\\
&\qquad G_2(\boldsymbol{X}) = 1 - \frac{(X_1+X_2-5)^2}{30} + \frac{(X_1-X_2-12)^2}{120}\\
&\qquad G_3(\boldsymbol{X}) = 1 - \frac{80}{(X_1^2+8X_2+5)}
\end{aligned}
$$

In NRBDO, we obtain 50 random samples from Eq. (10). As explained above, if the number of samples is small, we cannot assure that the reliability of a system is accurate. Therefore, the uncertainty of the reliability must be considered due to limited information.

Results of NRBDO are summarized and illustrated in Table 2 and Figure 1, respectively. For the cases 1, 2 and 3, from the point of view of DDO, these optimum points are in infeasible region since the first target reliability is 0.1. For cases 2, 5 and 8, the second target reliability is 0.5, so results of these cases are similar to those of RBDO. For cases 4, 5 and 6, the first target reliability is 0.5 and the second target reliabilities are different. For case 4, optimum point is in infeasible region because the second reliability guarantees probability of 0.1. However, for case 6, optimum point is in feasible region because the second reliability guarantees probability of 0.9. From the result of optimum point, we can show the trend with respect to the first and second target reliabilities. Note that as the target reliability increases, objective value also increases.

Results of DDO, RBDO and NRBDO are shown in Table 3 with the first and second reliabilities of 0.5. When distribution of input variables is symmetric and the first target reliability is 0.5, RBDO result is similar to DDO result. However, in RBDO, since we do not consider the number of samples to estimate the distribution of input variables, we cannot quantify the uncertainty of the reliability. In NRBDO, since we consider the number of samples and the second target reliability as 0.5, NRBDO result is similar to RBDO result. So these result shows that the proposed method is slightly more feasible than other methods.

Table 1. Comparison of RBDO results with respect to $\boldsymbol{R_1^{\text{target}}}$

R_1^{target}	0.1	0.5	0.9
x_1	8.7014	7.7907	6.9316
x_2	0.8139	1.7866	2.5821
$f(\boldsymbol{x})$	-7.8874	-6.0041	-4.3495

Table 2. Comparison of NRBDO results with respect to R_1^{target} and R_2^{target}

Case	R_1^{target}	R_2^{target}	x_1	x_2	$f(x)$
1	0.1	0.1	8.9747	0.5026	-8.4721
2	0.1	0.5	8.7058	0.8054	-7.9003
3	0.1	0.9	8.4971	1.0322	-7.4640
4	0.5	0.1	7.9318	1.6221	-6.3096
5	0.5	0.5	7.7672	1.7870	-5.9802
6	0.5	0.9	7.6049	1.9475	-5.6573
7	0.9	0.1	7.0807	2.4582	-4.6225
8	0.9	0.5	6.9041	2.6318	-4.2723
9	0.9	0.9	6.6894	2.8483	-3.8411

Table 3. Comparison of results of DDO, RBDO and NRBDO

	DDO	RBDO	NRBDO
n_t	NA	10^6	50
R_1^{target}	NA	0.5	0.5
R_2^{target}	NA	NA	0.5
x_1	7.7883	7.7907	7.7672
x_2	1.7928	1.7866	1.7870
$f(\boldsymbol{x})$	-5.9955	-6.0041	-5.9802

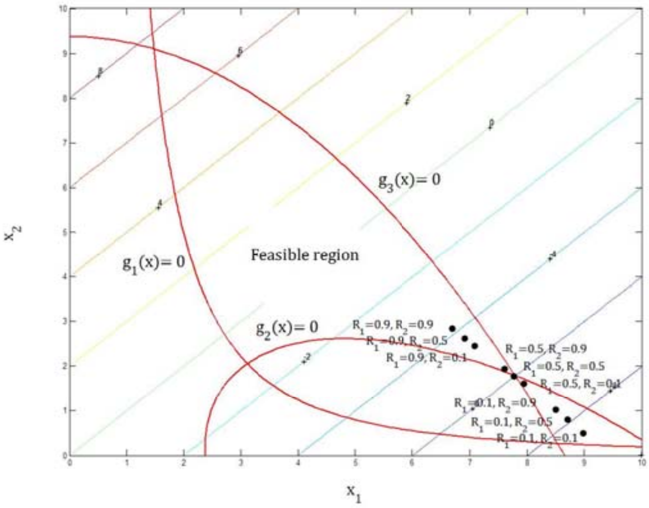

Figure 1. Nonparametric reliability-based design optimization result

Conclusions

In practical problems, the information of variables is often limited and discrete. However, recent studies of RBDO have to assume that the information of the input variables has a continuous distribution. Thus, the reliability estimated using recent methods has some uncertainty if we have only limited discrete information of the input variables. When we have limited discrete information, the uncertainty of the reliability as well as the reliability of a system should be considered. In order to consider the uncertainty of the reliability, we proposed a new method of NRBDO.

We consider the reliability of a system and the uncertainty of the reliability as the first and second reliabilities, respectively. To estimate the first reliability, we adopt AIC. In AIC, we use six types of candidate distributions, and then we select the best estimated distribution by maximizing the likelihood function of each candidate distribution. Using the best estimated distribution, we estimate the first reliability. To treat the uncertainty of the reliability, we perform nonparametric hypothesis test using sign test. In the sign test, the rejection region is calculated from binomial probability mass function.

To verify the proposed method, we use a mathematical example. Firstly, deterministic design optimization is performed without stochastic information of input variables. Secondly, RBDO is performed with respect to the first target reliability of 0.1, 0.5 and 0.9, respectively. The optimum point of RBDO with the first target reliability of 0.5 converges to DDO result. Finally, NRBDO is performed with respect to the first and second target reliabilities of 0.1, 0.5 and 0.9. The NRBDO result converges to RBDO and DDO results if the first and second target reliabilities are 0.5. The result shows that the optimum point tends to exist in infeasible region if the first target reliability is less than half. Also, the result shows that if the second target reliability is less than half and the first target reliability is 0.5, the optimum point tends to exist in infeasible region. The proposed method finds the optimum point by considering the reliability of a system as well as the uncertainty of the reliability when the information of the input variables is given in a limited discrete form.

Acknowledgments

This research was supported by the fundamental research subject (a study of the integrated framework for virtual design and analysis, no. GS-2) supervised by the ADD (Agency for Defense Development).

References

Akaike H. (1973), Information theory and an extension of the maximum likelihood principle. *Proceedings of the Second International Symposium on Information Theory*, pp. 267–281.

Al-Rubaie KS, Godefroid LB, Lopes JAM. (2007), Statistical modeling of fatigue crack growth rate in lnconel alloy 600. *I J Fatigue*. 29(5), pp. 931–940.

Breitung K. (1984), Asymptotic approximations for multinormal integrals. *ASCE J Eng Mech*. 110(3), pp. 357–366.

Conover WJ. (1980), *Practical nonparametric statistics 2nd*. New York: John Wiley & Sons.

Cornell CA. (1969), A probability-based structural code. *Journal of the American Concrete Institute*. 66(12), pp. 974–985.

Go SJ, Lee MC, Park MK. (2001), Fuzzy sliding mode control of a polishing robot based on genetic algorithm. *J Mech Sci Tech*. 15(5), pp. 580–591.

Hurvich CM, Simonoff JS, Tsai CL. (1998), Smoothing parameter selection in nonparametric regression using an improved Akaike informaion criterion. *J Royal Statistical Society Series B-Statistical Methodology*. 60, pp. 271–293.

Lee I, Choi KK, Noh Y. (2013), Comparison study between probabilistic and possibilistic methods for problems under a lack of correlated input statistical information. *Struct Multidiscipl Optim*. 47(2), pp. 175–189.

Lee SH, Kwak BM. (2006), Response surface augmented moment method for efficient reliability analysis. *Struct Saf*. 28(3), pp. 261–272.

Lim W, Lee TH. (2012), Reliability-based design optimization using Akaike information criterion for discrete information. *Trans Korean Soc Mech Eng A*. 36(8), pp. 921–927.

Noh Y, Choi KK, Lee I, Gorsich D, Lamb D. (2011), Reliability-based design optimization with confidence level under input model uncertainty due to limited test data. *Struct Multidiscipl Optim*. 43(4), pp. 443–458.

Pan W. (2001), Akaike's information criterion in generalized estimating equations. *Biometrics*. 57(1), pp. 120–125.

Rahman S, Xu H. (2004), A univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics. *Probab Eng Mech*. 19(4), pp. 393–408.

Sakamoto Y, Ishiguro M, Kitagawa G. (1986), *Akaike information criterion statistics*. Tokyo: KTK Scientific Publishers.

Shan S, Wang GG. (2007), Reliable design space and complete single-loop reliability-based design optimization. *Rel Eng Sys Saf*. 93(8), pp. 1218–1230.

Shetty NK, Soares CG, Thoft-Christensen P, Jensen FM. (1998), Fire safety assessment and optimal design of passive fire protection for offshore structures. *Rel Eng Sys Saf*. 61(1-2), pp. 139–149.

Spendelow JA, Nichols JD, Nisbet ICT, Hays H, Cormons GD, Burger J, Safina C, Hines JE, Gochfeld M. (1995), Estimating annual survival and movement rates of adults within a metapopulation of roseate terns. *Ecology*. 76(8), pp. 2415–2428.

Youn BD, Choi KK. (2004), An investigation of nonlinearity of reliability-based design optimization approaches. *J Mech Des*. 126(3), pp. 403–411.

Youn BD, Wang P. (2008), Bayesian reliability-based design optimization using eigenvector dimension reduction (EDR) method. *Struct Multidiscipl Optim*. 36(2), pp. 107–123.