A Finite Element Formulation of Minimization Problem for Steady Plastic Cycling

D.A. Tereshin

Department of Applied Mechanics, Dynamics and Strength of Machines, South-Ural State University, Lenin Avenue 76, Chelyabinsk City, Russia

denisat75@gmail.com

Abstract

Direct approach was extended beyond the scope of elastic shakedown to determine the parameters (strain increments and stress state) of an arbitrary steady elasto-plastic cycle under prescribed loading in ratcheting, alternating flow or in combination of the both. The resulting convex constrained optimization problem was formulated making use of finite element discretization. The object function equals to the work of fictitious elastic stresses on the plastic strains subtracted from the energy dissipation, with both terms being integrated over the cycle time and spatially over the body. The constraint system includes equality constraints of plastic strain incompressibility, cycle closure and initial residual stress self-balance, which are enforced by means of quadratic penalties; and inequality ones ensuring that the total stress is admissible. For testing purposes the Bree problem was solved employing the optimization formulation, with the resulting plastic strains over each half-cycle agreeing well with the analytical solution to the problem, even though a cycle was discretized only by two time instants.

Keywords: Direct computation, Finite elements, Steady cycle, Plastic ratcheting, Inelastic shakedown

Introduction

Elastic shakedown theory can be thought of as an extension of limit state theorems. The former allows determining the boundary between zone *II* of the Bree diagram (Bree 1967), shown in Fig. 1, and zones *III*–*V*. The further extension is the theory of steady inelastic structural response under repeated loadings (the response can be alternating flow – zone *III*, plastic ratcheting – zone *V* or the combination of the both – *IV*). Whereas in zone *II* of the Bree diagram a structure usually shakes down to purely elastic deformation over a relatively small number of cycles, stabilization to a steady cycle in excess of shakedown can proceed quite slowly and take many tens of cycles, with the structural behavior in steady condition being crucial for structural life assessments. The computations over loading history show that a slow stabilization usually occurs in severe alternating flow combined with incremental collapse (Abramov, Gadenin et al, 2011), which corresponds to the zone *IV* of in Fig. 1. In such case the whole stress-strain history can be very hard to compute using the step-by-step approach, with long loading history – not the amount of the degree of freedom unknowns – becoming the main trouble. This fact necessitates the development of an approach capable to find the steady cycle parameters (strain range and increment) directly. Although effective procedures for elastic shakedown have already been proposed and implemented, there are only few approaches being under development for steady plastic response, such as the algorithm by Chen and Ponter (2001), which is capable to capture the plastic shakedown boundary between zones *III* and *IV*, and the technique by

Spiliopoulos and Panagiotou (2012) employing the residual stress decomposition method to determine response for any kind of steady cycle.

In order to find the solution to a steady cycle problem, these approaches iteratively adjust either elastic constants or residual stress field. On the other hand, the problem of determining steady plastic response under prescribed cyclic loading can also be formulated as a convex constrained optimization problem (Gokhfeld and Cherniavsky, 1980). So the aim of the present study is to extend a mathematical optimization approach combined with finite element discretization to steady cycle problem so as to steady cycle state parameters for any region of the Bree diagram could be obtained.

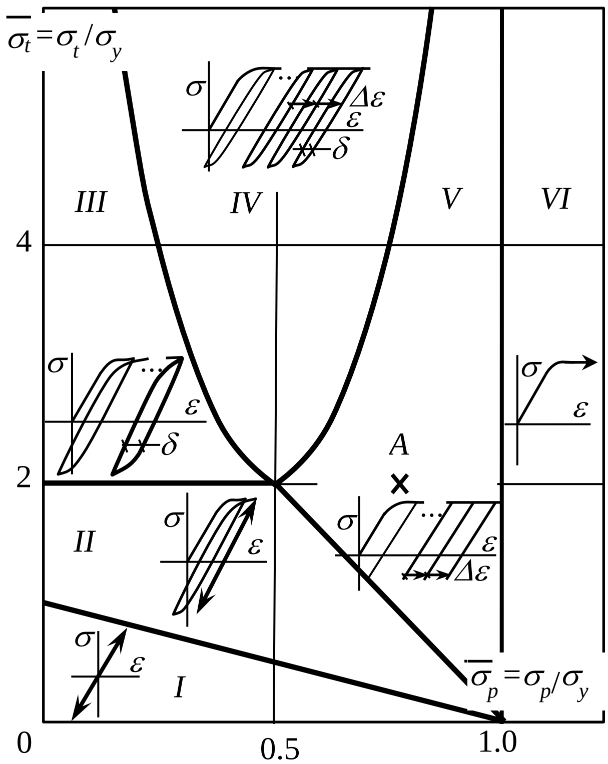

Figure 1. Bree problem

Mathematical optimization problem formulation

The system of expressions describing the deformation process in a steady state in terms of continuum mechanics can be written in the following way. The total stress is constituted by the fictitious elastic $\boldsymbol{\sigma}^e$ and residual $\boldsymbol{\rho}$ stresses, with the latter including the initial residual stresses $\boldsymbol{\rho}_0$ at the beginning of a cycle and the residual stress increment accumulated since then over time τ:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^e + \boldsymbol{\rho} = \boldsymbol{\sigma}^e + \boldsymbol{\rho}_0 + \int_0^{\tau} \dot{\boldsymbol{\rho}} d\xi \,. \tag{1}$$

The total strain rate is constituted by the elastic strain rate $\dot{\boldsymbol{\varepsilon}}^e$ part, which is caused by the change in the fictitious elastic stress $\boldsymbol{\sigma}^e$, and the parts due to the residual stress rate $\dot{\boldsymbol{\rho}}$ and the plastic strain rate $\dot{\boldsymbol{\varepsilon}}''$:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \mathbf{E}^{-1} \cdot \cdot \dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\varepsilon}}'' , \tag{2}$$

with **E** standing for the elastic stiffness matrix of the material.
The equilibrium for the fictitious elastic stress $\boldsymbol{\sigma}^e$ holds naturally. For the residual stress field $\boldsymbol{\rho}$, equilibrium over a body and at the part of its surface S_p with prescribed tractions can be stated as follows:

$$\nabla \cdot \boldsymbol{\rho}_0 = \mathbf{0} \text{ in } V, \qquad \boldsymbol{\rho}_0 \cdot \bar{n} = \mathbf{0} \text{ at } S_p; \tag{3}$$

$$\nabla \cdot \dot{\boldsymbol{\rho}} = \mathbf{0} \text{ in } V, \qquad \dot{\boldsymbol{\rho}} \cdot \bar{n} = \mathbf{0} \text{ at } S_p. \tag{4}$$

Since the total strain rate and the elastic rate fields in Eq. (2) are compatible, the plastic strain rate field and the strain rate field caused by the residual stresses are also compatible to a displacement rate $\dot{\mathbf{u}}$ field at every time moment:

$$\dot{\boldsymbol{\varepsilon}}'' + \mathbf{E}^{-1}\cdot\cdot\dot{\boldsymbol{\rho}} = \frac{1}{2}(\nabla\dot{\mathbf{u}} + \dot{\mathbf{u}}\nabla). \tag{5}$$

At each time instant in every material point the total stress is admissible: it lies inside or at the yield surface defined through a yield function $f(\boldsymbol{\sigma})$:

$$f(\boldsymbol{\sigma}) \le 0. \tag{6}$$

The associated flow rule holds for plastic strain with a non-negative plastic multiplier $\alpha \ge 0$:

$$\dot{\boldsymbol{\varepsilon}}'' = \alpha\nabla f(\boldsymbol{\sigma}), \tag{7a}$$

$$\alpha f(\boldsymbol{\sigma}) = 0, \tag{7b}$$

$$\alpha \dot{f}(\boldsymbol{\sigma}) = 0. \tag{7c}$$

Steady cycle condition may be enforced either in terms of residual stresses by making them exactly repeat every steady cycle $\boldsymbol{\rho}(\tau) = \boldsymbol{\rho}(\tau + T)$, with T – cycle time period:

$$\Delta\boldsymbol{\rho}(T) = \int_0^T \dot{\boldsymbol{\rho}}\,d\tau = 0, \tag{8a}$$

or, equivalently, in terms of strains by ensuring the plastic strain field increment $\Delta\boldsymbol{\varepsilon}''$ over a cycle to be compatible with displacement field increment $\Delta\mathbf{u}$:

$$\Delta\boldsymbol{\varepsilon}''(T) = \frac{1}{2}(\nabla(\Delta\mathbf{u}) + (\Delta\mathbf{u})\nabla), \;\; \Delta\boldsymbol{\varepsilon}''(T) = \int_0^T \dot{\boldsymbol{\varepsilon}}''\,d\tau. \tag{8b}$$

Making use of Drucker's postulate, the system of equations and constraints (1–8) may be recast according to (Gokhfeld and Cherniavsky, 1980) to a convex mathematical optimization problem by retaining Eqs. (1–6,8) as a system of equality and inequality constraints and searching for the minimum of the functional J, which can be proved to be zero at the exact minimizer, in the optimization field variables $\dot{\boldsymbol{\varepsilon}}''$, $\boldsymbol{\rho}$ and $\mathbf{u}$:

$$J = \int_0^T d\tau \int_V \left(\bar{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^e\right)\cdot\cdot\dot{\boldsymbol{\varepsilon}}''\,dV, \tag{9}$$

with $\dot{\boldsymbol{\varepsilon}}''$ associated to $\bar{\boldsymbol{\sigma}}$ by the flow rule (7a,b).

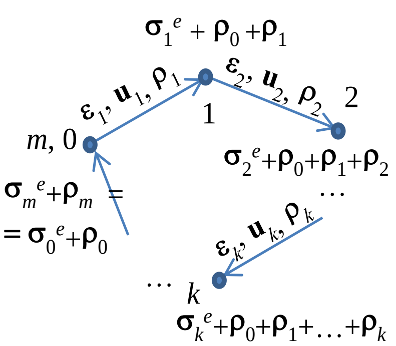

Figure 2. Time discretization

Let us make time discretization, as Fig. 2 shows, where time instant k is in the range from the beginning of a cycle ($k = 0$) to its end ($k = m$), with the both coinciding by the fact the cycle is closed. The strain $\boldsymbol{\varepsilon}_k$, displacement $\mathbf{u}_k$, residual stress $\boldsymbol{\rho}_k$ ($k = \overline{1,m}$) have the meaning of increment, $\boldsymbol{\rho}_0$ is initial residual stress, and the elastic stress $\boldsymbol{\sigma}_k^e$ corresponds to the k-th instant.

Employing finite element spatial discretization, the self-equilibrium condition (3) for the initial residual stress can be formulated as:

$$\mathbf{B}^T \boldsymbol{\rho}_0 \equiv \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \boldsymbol{\rho}_{0i} = \mathbf{0}, \tag{10a}$$

where NG is the total number of the Gauss integration points, w_i is the integration weight of point i. From now on all vectors and matrices indexed by i or j imply local ones for the i-th or j-th point, whereas not indexed matrices – global ones. E.g., Hook's relation between the stress and strain for point i at time instant k is $\boldsymbol{\sigma}_{ik}^e = \mathbf{E}_i \boldsymbol{\varepsilon}_{ik}^e$, with $\boldsymbol{\varepsilon}_{ik}{}^T = \{\varepsilon_x^{ik} \quad \varepsilon_y^{ik} \quad \varepsilon_z^{ik} \quad 2\varepsilon_{xy}^{ik} \quad 2\varepsilon_{yz}^{ik} \quad 2\varepsilon_{zx}^{ik}\}$, but the global relation is $\boldsymbol{\sigma}_k^e = \mathbf{E}\boldsymbol{\varepsilon}_k^e$.

Having plastic strain increment $\boldsymbol{\varepsilon}_k^{''}$, in place of Eq. (5) one obtains the displacements induced $\mathbf{u}_k$ by solving for the FE problem with global stiffness $\mathbf{K}$ and deformation $\mathbf{B}$ matrices:

$$\mathbf{u}_k = \mathbf{K}^{-1} \int_V \mathbf{B}^T \mathbf{E} dV \boldsymbol{\varepsilon}_k^{''}, \tag{10b}$$

and then restores the residual stresses:

$$\boldsymbol{\rho}_k = \mathbf{E}\left(\mathbf{B}\mathbf{u}_k - \boldsymbol{\varepsilon}_k^{''}\right). \tag{10c}$$

The total stress (1) evolves through the cycle points as Fig. 2 illustrates:

$$\boldsymbol{\sigma}_{ik} = \boldsymbol{\sigma}_{ik}^e + \boldsymbol{\rho}_{i0} + \sum_{l=1}^{k} \boldsymbol{\rho}_{il} \quad . \tag{10d}$$

The cycle closure condition (8a) takes the form of

$$\sum_{k=1}^{m} \boldsymbol{\rho}_{ik} = \mathbf{0}, \tag{10e}$$

and the stress admissibility condition (6) is required to be satisfied for $\forall k = \overline{1,m}$, $i = \overline{1,NG}$:

$$f(\boldsymbol{\sigma}_{ik}) \le 0. \tag{10f}$$

Accepting von Mises yield condition the functional J becomes:

$$J = \sum_{k=1}^{m} \sum_{i=1}^{NG} \left[\sqrt{2} w_i \tau_y \sqrt{\boldsymbol{\varepsilon}_{ik}^{''T} \mathbf{D}_i \boldsymbol{\varepsilon}_{ik}^{''}} - w_i \boldsymbol{\sigma}_{ik}^{e\,T} \boldsymbol{\varepsilon}_{ik}^{''} \right], \tag{10g}$$

where τ_y is the yield shear stress, $\mathbf{D}_i = Diag\{1 \quad 1 \quad 1 \quad 1/2 \quad 1/2 \quad 1/2\}$.

Thus, the statement of mathematical optimization problem becomes: $\underset{\boldsymbol{\varepsilon}'',\boldsymbol{\rho},\mathbf{u}}{\text{minimize}}\, J$ subject to Eqs. (10a–f) satisfied for $\forall k = \overline{1,m}, \forall i = \overline{1,NG}$, in addition to which, one has to impose incompressibility constraints on plastic strains:

$$\mathbf{D}_{Vi} \boldsymbol{\varepsilon}_{ik}^{''} = \mathbf{0}, \; \forall k = \overline{1,m}, \; \forall i = \overline{1,NG}, \tag{10h}$$

where $\mathbf{D}_{Vi}$ is the diad projecting to the spherical part of tensor: $\mathbf{D}_{Vi} = \frac{1}{3}(1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0)^T (1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0)$.

In order to eliminate displacements, by substitution Eq. (10b) to Eq. (10c) one gets to:

$$\boldsymbol{\rho}_k = -\mathbf{EP}\boldsymbol{\varepsilon}''_k,\ \forall k = \overline{1,m} \tag{11}$$

with the matrix $\mathbf{P} = \mathbf{I} - \mathbf{BK}^{-1}\int_V \mathbf{B}^T \mathbf{E} dV$ projecting an arbitrary strain to the subspace of self-equilibrated strains.

For convenience's sake, let us accept the notation from Vu and Yan et al (2004), which simplifies energy relations: $\mathbf{e}_{ik} = w_i \mathbf{D}^{1/2}\boldsymbol{\varepsilon}_{ik}$ – for strain increment vector; $\hat{\mathbf{B}}_i = w_i \mathbf{D}_i^{1/2}\mathbf{B}_i$ – for deformation matrix, so that $\sum_{k=1}^{m}\mathbf{e}_{ik} = \hat{\mathbf{B}}_i \mathbf{u}$; $\mathbf{t}_{ik} = \mathbf{D}_i^{-1/2}\boldsymbol{\sigma}_{ik}^e$ and $\boldsymbol{\beta}_{ik} = \mathbf{D}_i^{-1/2}\boldsymbol{\rho}_{ik}$ – for stress vectors. In these definitions $\mathbf{D}_i^{-1/2}$ and $\mathbf{D}_i^{1/2}$ are diagonal symmetric matrices such that: $\mathbf{D}_i^{-1/2} = (\mathbf{D}_i^{1/2})^{-1}$ and $\mathbf{D}_i = \mathbf{D}_i^{1/2}\mathbf{D}_i^{1/2}$.

Von Mises yield condition (10f) can be presented in the form of Euclidian norm:

$$\left\| (\mathbf{I}_i - \mathbf{D}_{Vi})\left(\mathbf{t}_{ik} + \boldsymbol{\beta}_{i0} + \sum_{l=1}^{k}\boldsymbol{\beta}_{il} \right) \right\| \le \sqrt{2}\tau_y . \tag{12}$$

Let us define the matrix $\mathbf{G}$ a component $\mathbf{G}_{ij}$ of which relates plastic strain at point j to the induced residual stress at point i:

$$\boldsymbol{\beta}_{ik} = \mathbf{G}_{ij}\mathbf{e}_{jk},\ \forall k = \overline{1,m},\ \forall i, j = \overline{1,NG}, \tag{13}$$

where $\mathbf{G}_{ij} = \mathbf{D}_i^{-1}\mathbf{E}_{\_wi}\hat{\mathbf{B}}_i \mathbf{K}^{-1}\hat{\mathbf{B}}_j^T \mathbf{E}_{\_wj}\mathbf{D}_j^{-1} - \mathbf{D}_i^{-1/2}\mathbf{E}_{\_wi}\mathbf{D}_j^{-1/2}\delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \ne j$; $\mathbf{E}_{\_w_i} = (1/w_i)\mathbf{E}_i$). In the new terms the optimization problem simplifies to:

$$\underset{\mathbf{e}_{ik},\,\boldsymbol{\beta}_{i0}}{\text{minimize}} \sum_{i=1}^{NG}\sum_{k=1}^{m}\left(\sqrt{2}\tau_y \sqrt{\mathbf{e}_{ik}{}^T\mathbf{e}_{ik} + \varepsilon_0{}^2} - \mathbf{t}_{ik}{}^T\mathbf{e}_{ik} \right) \tag{14a}$$

$$s.t.\left|\begin{array}{ll}
\hat{\mathbf{B}}_i{}^T \boldsymbol{\beta}_{i0} = \mathbf{0}, & \text{(14b)}\\
\sum_{j=1}^{NG}\mathbf{G}_{ij}\sum_{k=1}^{m}\mathbf{e}_{jk} = \mathbf{0}, & \text{(14c)}\\
\mathbf{D}_{\mathrm{V}i}\mathbf{e}_{ik} = \mathbf{0}, & \text{(14d)}\\
f_{ik} = \left(\boldsymbol{\beta}_{i0} + \sum_{j=1}^{NG}\mathbf{G}_{ij}\sum_{l=1}^{k}\mathbf{e}_{jl} + \mathbf{t}_{ik} \right)^T (\mathbf{I}_i - \mathbf{D}_{Vi})\left(\boldsymbol{\beta}_{i0} + \sum_{j=1}^{NG}\mathbf{G}_{ij}\sum_{l=1}^{k}\mathbf{e}_{jl} + \mathbf{t}_{ik} \right) - 2\tau_y{}^2 \le 0, & \text{(14e)}
\end{array}\right.$$

for $\forall i = \overline{1,NG},\ k = \overline{1,m}$.

According to Vu and Yan et al (2004) a small regularization parameter ε_0 was introduced to Eq. (14a) in order to make it differentiable at $\mathbf{e}_{ik} = \mathbf{0}$.

Test problem

Since the Bree problem of pressurized thin-walled tube under repeated thermal loading has an analytical solution (Bree, 1967), it was used to prove the optimization

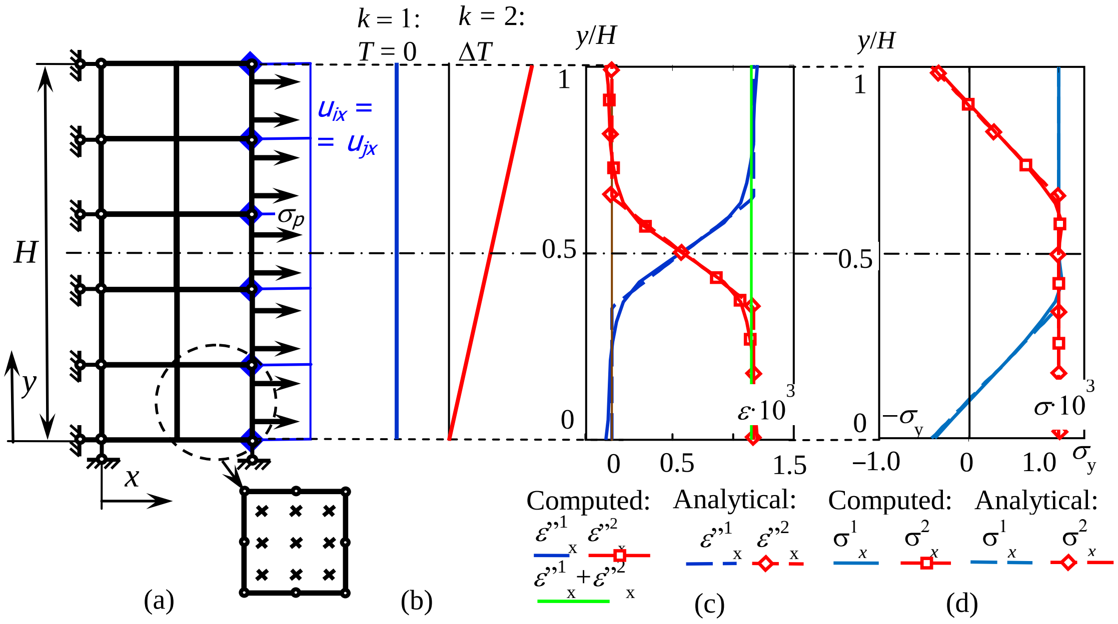

Figure 3. Problem sketch (a), temperature distribution (b), a comparison of computed and analytical plastic strains (c) and total stresses (d)

formulation (14) is capable of yielding the solution. If Tresca yield criterion is implied, the Bree problem is equivalent to the plain stress problem corresponding to a cross section of the tube under the same time-varying thermal stress σ_t and the constant stress σ_p equal to the hoop stress in the former problem caused by pressure. The structure was discretized with rectangular quadratic finite elements as Fig. 3a shows (0*y* points along a radial direction, 0*x* – in a hoop one), each having nine integration points. The u_x displacement at the left edge is restrained, and all the nodes at the right edge have an identical value of u_x. In a cycle, temperature varies from uniform distribution at time point $k = 1$ to a linear distribution with the gradient directed along 0*y* axis at the end of a cycle ($k = 2$) (see Fig. 3b). The constant applied stress σ_p amounts to $0.75\sigma_y$, the maximal thermal stress $\sigma_t = 2\sigma_y$. The corresponding state is depicted by point *A* in the Bree diagram Fig. 1, at which pure incremental collapse is expected.

Solution technique

The conditions of plastic incompressibility (14d), cycle closure (14c) and initial residual stress self-balance (14b) are enforced by means of quadratic penalties in an extended object function:

$$f_0 = \sum_{i=1}^{NG}\sum_{k=1}^{m}\left(\sqrt{2}\tau_y\sqrt{\mathbf{e}_{ik}{}^T\mathbf{e}_{ik}+\varepsilon_0{}^2}-\mathbf{t}_{ik}{}^T\mathbf{e}_{ik}\right)+\frac{c_1}{2}E\sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{e}_{ik}{}^T\mathbf{D}_{Vi}\mathbf{e}_{ik}+$$
$$+\frac{c_2}{2E}\sum_{i=1}^{NG}\sum_{j=1}^{NG}\left(\sum_{l=1}^{m}(\mathbf{e}_{il})^T\left(\sum_{g=1}^{NG}\mathbf{G}_{gi}{}^T\mathbf{G}_{gj}\right)\sum_{k=1}^{m}(\mathbf{e}_{jk})\right)+\frac{c_3}{2EL_e}\sum_{i=1}^{NG}\sum_{j=1}^{NG}\left(\boldsymbol{\beta}_{i0}{}^T\hat{\mathbf{B}}_i\hat{\mathbf{B}}_j{}^T\boldsymbol{\beta}_{j0}\right), \quad (15)$$

where c_1,c_2,c_3 are penalty coefficients, L_e – typical element size.

An unconstrained mathematical optimization problem is formulated using logarithmic barrier functions for the inequality constraints (14e):

$$\underset{\mathbf{e}_{ik},\,\boldsymbol{\beta}_{i0}}{\text{minimize}}\quad F_0, \quad (16)$$

in which $F_0 = f_0 - \frac{1}{t}\sum_{i=1}^{NG}\sum_{k=1}^{m}\log(-f_{ik})$.

Having denoted $\mathbf{f}$ – the vector constituted by all the inequality constraint functions (14e), one obtains the stationary condition of F_0 as central path conditions:

$$\begin{aligned} &\nabla f_0 + \sum_{i=1}^{NG}\sum_{k=1}^{m}\lambda_{ik}\nabla f_{ik} = \mathbf{0}, \\ &-\mathbf{diag}(\boldsymbol{\lambda})\mathbf{f} - (1/t)\mathbf{I} = \mathbf{0}, \end{aligned} \tag{17}$$

or, from another viewpoint, as the modified Karush-Khun-Tucker equations of the problem of minimization functional (15) subject to (14e) (Boyd and Vandenberghe, 2004), with the gradients being taken with respect to the primal variables ($\mathbf{e}$,$\boldsymbol{\beta}_0$), and $\boldsymbol{\lambda}$ being the dual variable vector. So solving Eqs. (17) one applies a simple primal-dual interior-point approach; and the greater the parameter t, the closer the solution of Eq. (17) to the solution to the original problem given by Eqs. (14a-e).

However, before solving Eqs. (17), one has to find a feasible point ($\mathbf{e}$,$\boldsymbol{\beta}_0$) satisfying all the inequality constraints (14e). This was done by means of a phase I method:

$$\begin{aligned} &\underset{\mathbf{e}_{ik},\,\boldsymbol{\beta}_{i0}}{\text{minimize}}\ S \\ &s.t.\left|\begin{aligned} &f_{ik}(\mathbf{e}_{ik},\boldsymbol{\beta}_{i0}) \le S,\ i=\overline{1,NG},\ k=\overline{1,m}, \\ &F_0 \le M. \end{aligned}\right. \end{aligned} \tag{18}$$

For initialization S was taken such as to strictly satisfy all the inequality constraints, and $M \approx 10|f_0(\mathbf{e}=\mathbf{0}, \boldsymbol{\beta}_0=\mathbf{0})|$. Having started with $\varepsilon_0 = 10^{-4}$, and small $c_i = 10^{-5}$ so as not to pay much of attention to the equality constraints of the original problem (Eqs. (14b-d)) at this stage, the phase I method converges quite rapidly as inequality constraint functions expressed by Eqs. (14e) are quadratic, and the algorithm based on Newton's method described below performs at quadratically convergent stage immediately.

Eqs. (17) were solved by means of Newton's method using the analytical expressions of first and second derivatives to get primal and dual variable increments (($\Delta\mathbf{e}$,$\Delta\boldsymbol{\beta}_0$) and $\Delta\boldsymbol{\lambda}$). At the second stage exact line search was employed, as without line search, Newton's method experiences convergence difficulties when $\|\mathbf{e}_{ik}\| \ge \varepsilon_0$. The line search uses the extended object function F_0 as a merit function and ensures $\boldsymbol{\lambda} \succ \mathbf{0}$ and the incremented value of ($\mathbf{e}$,$\boldsymbol{\beta}_0$,$\boldsymbol{\lambda}$) to be feasible. Basing on the value of surrogate duality gap $\eta = -\mathbf{f}^{\mathrm{T}}\boldsymbol{\lambda}$ the parameter t was determined in each iteration as: $t = \mu \cdot NG \cdot m/\eta$; the more μ is, the more aggressively t increases.

Convergence difficulties entailed by the sum of Euclidian norms term in the object function F_0 were remedied by the proper choice of μ value, combined with adjusting the regularization parameter ε_0 and the penalty coefficients c_i. In general, the object function reduces successfully when μ has a relatively small value of 1–5, and all of the four terms in Eq. (15) have nearly the same order of magnitude. However, approaching the boundary defined by inequality constraints Eqs. (14e), the method can drastically slow down (Wright, 1997). This obstacle was dealt with by a temporal reduction of μ for several iterations to about 0.05, after which the point becomes repelled enough form the boundary for the object function to be reduced further.

At the end of the convergence process, ε_0 was adjusted to be 10^{-5}–10^{-6} (2-3 order of magnitude less than maximal $\|\mathbf{e}_{ik}\|$), and the coefficients c_i amounted to 10^5–10^{10} for the constraints given by Eqs. (14b-d) to be fulfilled well.

Results

Three mesh patterns were used to solve the problem: 5 elements in column and 2 in row, as Fig. 3a depicts, 10×1 elements, and 5×1 elements. Only two instances were considered over cycle period ($m = 2$) correspondent to no temperature applied and to the full thermal load.

The convergence process and results obtained in different cases of mesh patterns are nearly the same. One has to admit that in spite of the fact that the penalty coefficients c_i were increased gradually, and the regularization coefficient ε_0 appearing in the sum of norms term was also reduced gradually, the convergence was neither fast nor stable since Newton's method is naturally not suited for the sum of norms problem. So it required about a hundred of iterations to converge.

Nevertheless, in spite of the fact that the cycle was discretized over time only by two time points, one can see a good agreement between the numerical solution (solid lines) and the analytical one (dashed lines) shown in Figs. 3c,d for strain ε_x" and total stress σ_x distributions along 0y axis (the stresses are normalized by the Young modulus E, e.g. the yield stress $\sigma_y = 0.001$). Figure 3c also shows that the strain increment over the cycle is compatible, and the checks performed confirm that the residual stress β_0 is self-balanced, and the plastic strain is deviatoric, which means the constraints presented by Eqs. 14b-d are satisfied.

Conclusion

It has been shown that the general problem of steady cycle can be solved directly by stating it as a convex mathematical optimization problem with the use of finite element discretization.

This formulation has been proved to be able, in principle, to capture the proper results for all the parameters of a steady cycle such as plastic strains and residual stresses, even though the simple computational approach implemented in the study was used for demonstration only and is not claimed to be efficient for the real problem.

References

Abramov A.V., Gadenin M.M., Makhutov N.A., Evropin S.V., Cherniavsky A.O. and Cherniavsky O.F. (2011), Low-cycle deformation and fracture of structures. *Handbook. An engineering journal*, 11, 24p. (in Russian)

Boyd S. and Vandenberghe L (2004), Convex optimization. *Cambridge University Press*.

Bree J. (1967), Elastic-plastic behavior of thin tubes subjected to internal pressure and intermittent high-heat fluxes with application to fast nuclear reactor fuel elements. *J. Strain Anal.*, 2(3), pp. 226-238.

Chen H. and Ponter A.R.S. (2001), A method for the evaluation of a ratchet limit and the amplitude of plastic strain for bodies subjected to cyclic loading. *Eur. J. Mech. A/Solids*, 20, pp. 555–571.

Gokhfeld D.A. and Cherniavsky O.F. Limit analysis of structures at thermal cycling. *Sijthoff and Noordhoff, Leyden*, 1980.

Spiliopoulos K.V. and Panagiotou K.D. (2012), A direct method to predict cyclic steady states of elastoplastic structures. *Comput. Methods Appl. Mech. Engrg.*, 223–224, pp. 186–198.

Vu D. K., Yan A. M. and Nguyen-Dang H. (2004), A primal-dual algorithm for shakedown analysis of structures. *Comput. Methods Appl. Mech. Engrg.*, 193, pp. 4663-4674.

Wright S.J. (1997), Primal-dual interior-point methods. *SIAM*.