Comparison of various numerical discretisation approaches for the scaled boundary method

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Abstract

The scaled boundary method is a semi-analytical method for solving linear partial differential equations. In the scaled boundary method, the discretisation approach used in the circumferential direction has significant influence on the accuracy of the resulting solutions. The most commonly used method for performing this circumferential discretisation is the finite element approach, leading to the method called the scaled boundary finite element method (SBFEM), and most previous work using the SBFEM has employed linear or quadratic isoparametric elements. In this paper, various alternative numerical discretisation approaches for the scaled boundary method developed by the authors are reviewed and compared, including higher-order finite elements, the meshless local Petrov-Galerkin approach, the Element-free Galerkin approach and Fourier shape functions. These approaches have significant advantages in accuracy and convergence compared with the conventional SBFEM with linear or quadratic elements. Numerical examples are provided to compare the above mentioned scaled boundary methods in terms of accuracy and convergence, and the performance of these various approaches in different cases will be discussed.

Keywords: Scaled boundary method, numerical discretisation approaches, computational accuracy, comparisons.

Introduction

The scaled boundary method is a semi-analytical method developed relatively recently by Wolf and Song (Wolf and Song, 1996; Wolf, 2003). The method introduces a normalised radial coordinate system based on a scaling centre and a defining curve (usually taken as the boundary). The governing differential equations are weakened in the circumferential direction and then solved analytically in the normalised radial direction. Like the boundary element method, discretisation of the boundary only is required, but unlike that method no fundamental solution is required. The method has been shown to be more efficient than the finite element method for problems involving unbounded domains and for problems involving stress singularities or discontinuities (Deeks and Wolf, 2002a).

The Figure 1 shows a typical bounded scaled boundary coordinate system with a scaling centre (x_0, y_0), radial coordinate ξ and circumferential coordinate *s*. An approximate solution for displacement is sought in the form

$$\left\{u_h(\xi,s)\right\} = \sum_{i=1}^{n}[N_i(s)]u_{hi}(\xi) = [N(s)]\left\{u_h(\xi)\right\} \tag{1}$$

This represents a discretisation of the part of the boundary located at $\xi = 1$ with the shape function $[N(s)]$. The unknown vector $\{u_h(\xi)\}$ is a set of n functions analytical in ξ. The shape functions apply for all lines with a constant ξ.

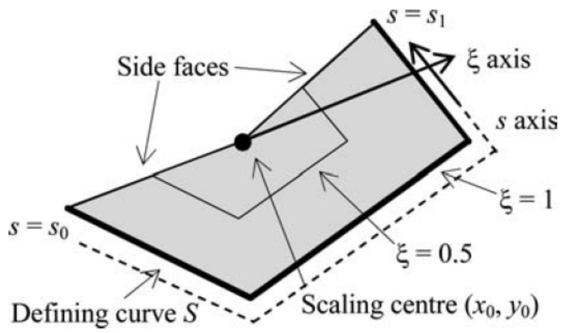


Figure 1 A scaled boundary coordinate system

In the scaled boundary method, the discretisation approach used in the circumferential direction has significant influence on the accuracy of the resulting solutions. The most commonly used method for performing this circumferential discretisation is the finite element approach, leading to the method called the scaled boundary finite element method (SBFEM) .

The scaled boundary method involves solution of a quadratic eigenproblem, the computational expense of which increases rapidly as the number of degrees of freedom increases. Therefore it is desirable to obtain solutions at a specified level of accuracy while using the minimum number of degrees of freedom necessary. Generally, there are two approaches to improve the accuracy of SBFEM. The first one is to use an adaptive approach to refining the discretisation of the boundary (Deeks and Wolf, 2002b). The second one is to use new shape functions, such as higher-order polynomial shape functions (Vu and Deeks, 2006, 2008a, 2008b) and meshless approaches (Deeks and Augarde, 2005; He et al., 2012) which may provide better performance in terms of convergence and smoothness of the solution.

This paper reviews various numerical discretisation approaches for the scaled boundary method. After a review of previous work in this field, the paper focuses on a comparison of the performance of these approaches and the conventional SBFEM through numerical examples. The accuracy and convergence of these approaches will be illustrated and compared, and discussions and conclusions will be provided.

Review

Higher –order elements

The use of higher-order elements (p>2, where p represents the polynomial order of the shape functions) into the SBFEM is presented by Vu and Deeks (Vu and Deeks, 2006). Two approaches are examined: the spectral element approach with Lagrange shape functions interpolating for an increasing number of internal nodes; and a p-hierarchical approach where polynomial shape functions forming a hierarchical basis are employed to add new DOFs into the domain without changing the existing ones. It was proved that the SBFEM converged significantly faster with higher-order elements than linear or quadratic elements, and the performance of the two different types of higher-order elements is very similar. In order to take advantage of higher-order elements, Vu and Deeks further developed the p-adaptive technique for SBFEM (Vu and Deeks, 2008a, 2008b) and demonstrated SBFEM converged significantly faster under p-refinement than under h-refinement.

Meshless approaches

The meshless methods provide alternative approaches for SBFEM. The meshless methods are only based on nodes, and thus no mesh generation or remeshing is required. It has been shown that, compared with the finite element method, the meshless method (for example, the Element-free Galerkin method) has the advantages of high accuracy, rapid convergence, and a smooth stress solution can be obtained without post-processing. Deeks and Augarde (Deeks and Augarde, 2005) developed a Meshless Local Petrov-Galerkin method scaled boundary method (MLPG-SBM) and He et al (He et al, 2012) developed an Element-free Galerkin scaled boundary method (EFG-SBM). These works showed that these two meshless scaled boundary methods gave a higher level of accuracy and rate of convergence than the conventional SBFEM using linear or quadratic elements.

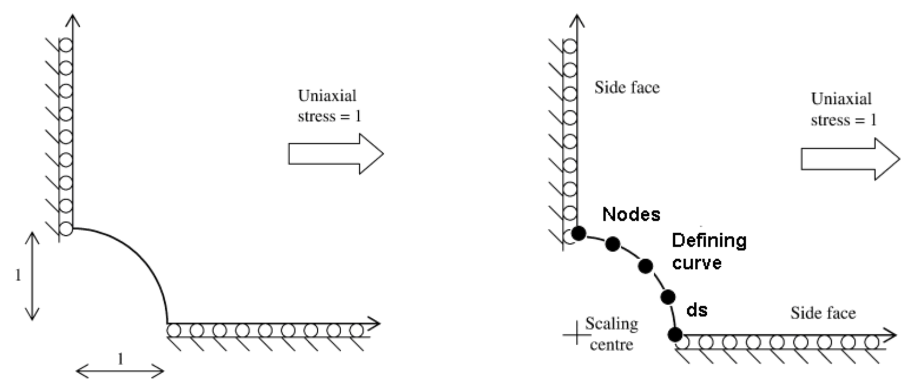
Fourier methods

Fourier interpolation containing trigonometric functions has been demonstrated to be more accurate than the classic Lagrange shape functions in various studies of the BEM. He et al (He et al, 2013) investigated the possibility of using Fourier shape functions in the SBFEM to form the approximation in the circumferential direction. The shape functions effectively form a Fourier series expansion in the circumferential direction, and are augmented by additional linear shape functions. By solving benchmark elastostatic and steady-state heat transfer problems, it was demonstrated that the accuracy and convergence of Fourier interpolation based scaled boundary method has remarkable advantages in computational accuracy and convergence.

Examples for comparison

Example 1 - A plate of infinite extent subjected to a uniaxial stress

The first example is a plate of infinite extent containing a hole of unit radius and subjected to a uniaxial stress of unity. Considering the symmetry of the problem, one quarter of the plate is represented (Figure 2(a)). The Young's modulus is taken as $E = 1000$ and Poisson's ratio as $v = 0.3$. The boundary of the hole is considered as a single edge with the length l (Figure 2(b)). Uniform nodes are introduced along this edge, the nodes spacing is specified as ds. The centre of the hole is used as the scaling centre.



(a) (b)

Figure 2 A plate of infinite extent subjected to a uniaxial stress: (a) geometry and (b) scaled boundary model.

In order to evaluate the computed error in the domain, the relative error in energy norm is employed. Figure 3 shows the variation of the relative error in energy norm with degrees of freedom (DOF) in the domain $\xi = [1, 3]$. It is clear that the EFG-SBM and MLPG-SBM with quadratic basis are more accurate than conventional quadratic SBFEM with the same DOF, but the rate of convergence is similar. However, the higher-order elements and Fourier shape functions achieve a more rapid rate of convergence, showing that higher accuracy can be obtained under the same number of DOF compared with the conventional SBFEM and meshless SBM.

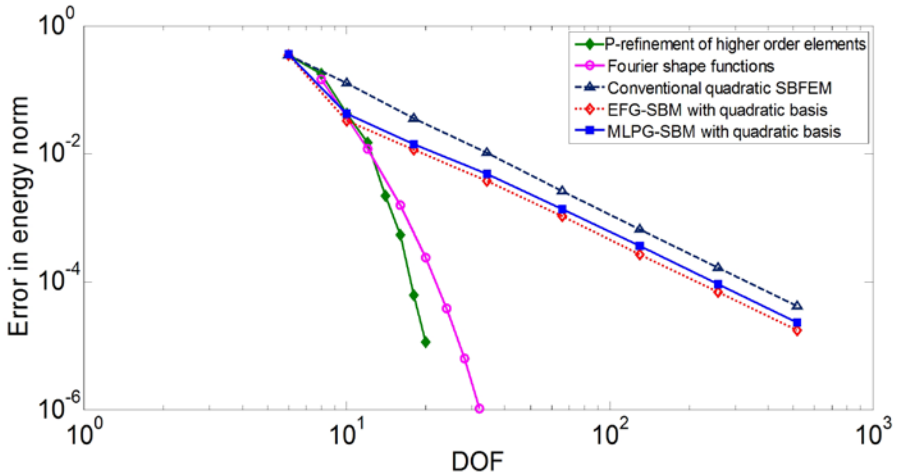
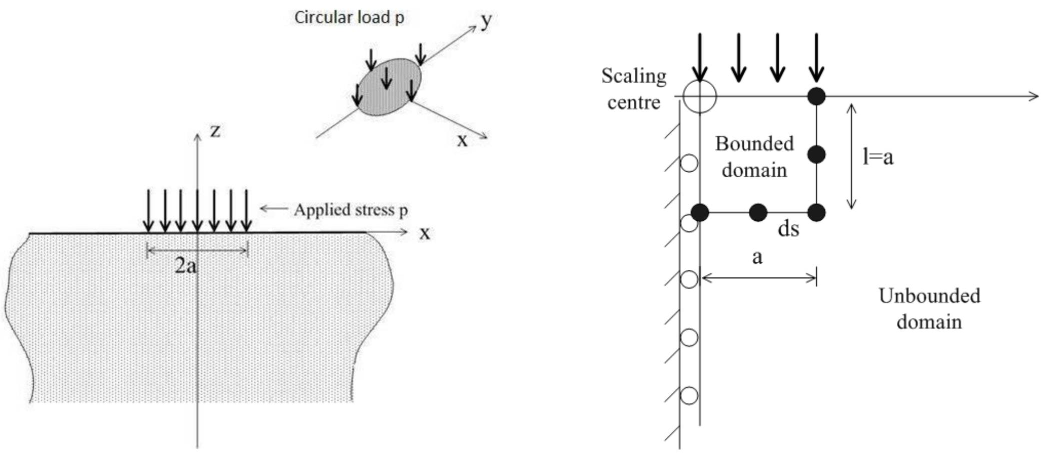


Figure 3 The variation of error in energy norm with degrees of freedom

Example 2 – Uniform circular load on homogeneous half-space

The second example is the problem of a uniform load on an elastic homogeneous half-space, as shown in Figure 4(a). A vertical pressure p_0 is applied uniformly over the circular region of radius a as shown. The Young's modulus is taken as $E = 1000$ and Poisson's ratio as $v = 0.3$. This problem is axisymmetric. The scaled boundary model is illustrated in Figure 4(b), where the axis of symmetry runs along the left hand side of the domain. Two sub-domains are employed, sharing a common scaling centre, one bounded and one unbounded. Uniform nodes are distributed on the defining curve separating the two sub-domains, with node spacing ds. The stiffness matrix for each sub-domain is formed independently using the scaled boundary approach, and then the two stiffness matrices are assembled together. This problem has an analytical solution.



(a) (b)

Figure 4 Uniform circle/strip load on homogeneous half-space: (a) geometry and (b) scaled boundary model.

In Figure 5, the variation of relative error for displacement on the central point (x=0,y=0,z=0) is illustrated. It is seen that the EFG-SBM has better accuracy than conventional SBFEM using the same order of basis functions or elements, but the higher-order elements are still seen to be have better convergence. The Fourier shape functions' performance is not quite so close to the higher order elements in this example.

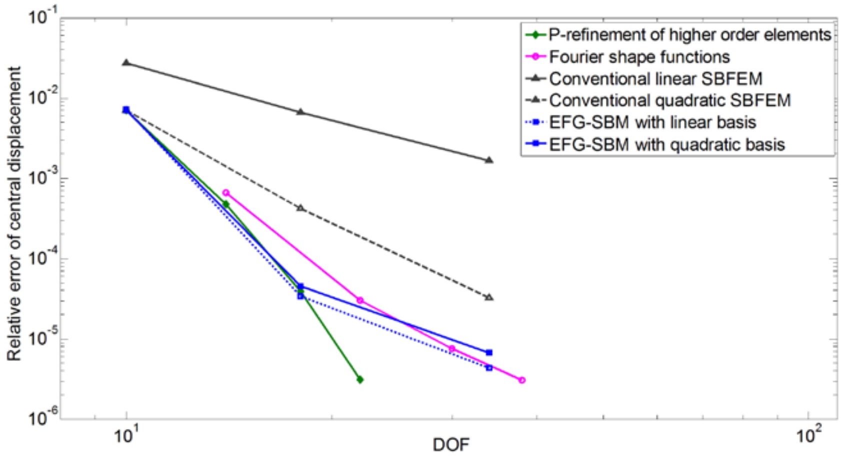


Figure 5 The variation of relative error for central displacement with degrees of freedom

Example 3 – Steady-state heat transfer in an L-shape domain

The third example deals with the steady-state heat transfer problem in an L-shaped domain illustrated in Figure 6(a). The exact solution for this example contains a thermal singularity at the re-entrant corner, and is given in the polar coordinate system with the origin O by

$$T(r,\theta) = r^{\frac{2}{3}} \sin(\frac{2\theta - \pi}{3}) \tag{2}$$

The specified boundary conditions are also indicated in Figure 6(a), with zero Dirichlet conditions at part of the boundary that lies on axis and Neumann conditions determined by the exact solution $T(r,\theta)$ elsewhere. The exact solution of heat flux on the radial direction q_r is

$$q_r = -k \cdot \frac{2}{3} r^{-\frac{1}{3}} \sin(\frac{2\theta - \pi}{3}) \tag{3}$$

where k is the thermal conductivity, taken as unit value in this example.

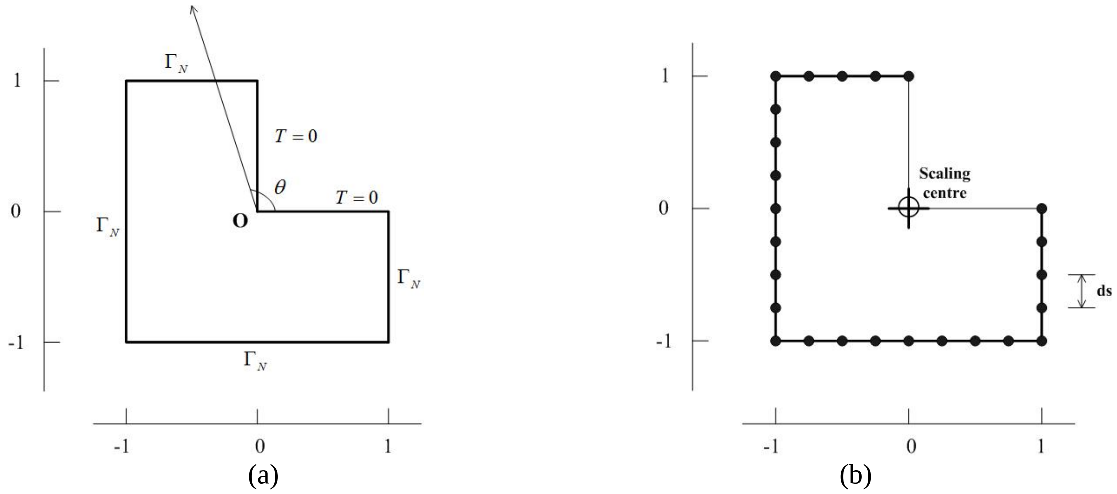


Figure 6 Steady-state heat transfer in an L-shape domain: (a) geometry and (b) scaled boundary model.

In order to evaluate the accuracy, the values of q_r are computed along the line with angle $\theta = \pi$, where ten points are located on radial coordinates $r = 0$ and average relative error on these points are obtained to compare the performance of different shape functions, as illustrated in Figure 7. It is clearly demonstrated that the higher-order shape functions and Fourier shape functions are converge much more rapidly than the EFG-SBM and conventional SBFEM, and it is also shown that higher accuracy is obtained using the EFG-SBM with a linear basis compared with the linear SBFEM.

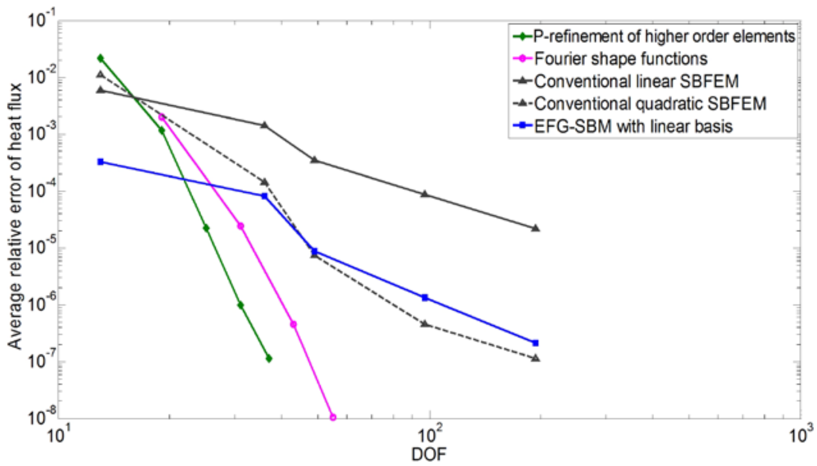


Figure 7 The variation of relative error for central displacement with degrees of freedom

Example 4 - Closed-form near-tip problem of an infinite plate with a through crack

The example is an infinite plate containing a through crack and subjected to uniaxial tension in a direction normal to the crack, as illustrated in Figure 8(a). This is a mode I crack problem. The closed-form near-tip problem will be solved to investigate the performance of the scaled boundary method with different shape functions in solving for the stress field in a bounded domain around the crack tip when subjected to prescribed boundary displacements obtained from the exact solution. Along the boundary of square region ABCD with size $\bar{a} = 0.01a$ in Figure 8(b), the closed-form near-tip displacements are imposed. The expression of the closed-form displacement and stress field can be found in. The scaled boundary model is shown in Figure 8(b). Uniform nodes with spacing ds are used.

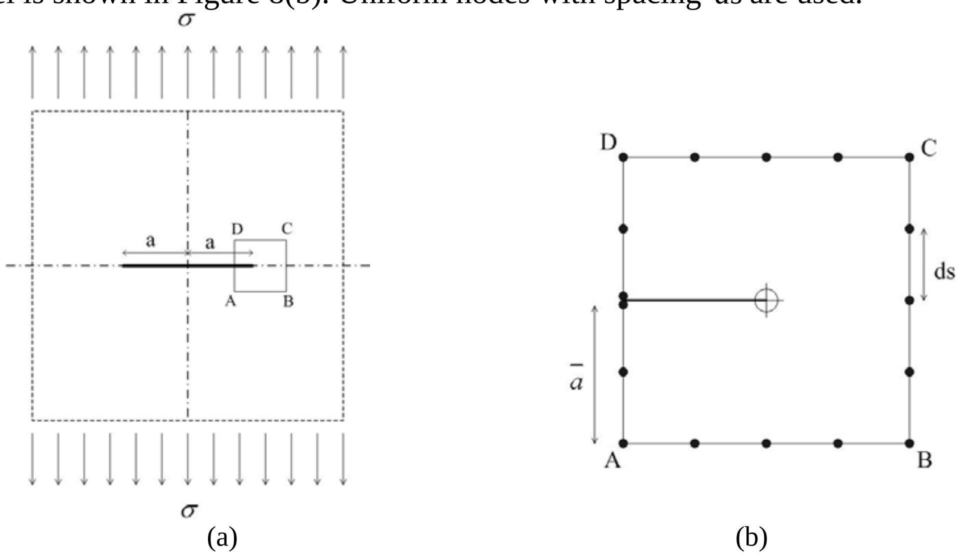


Figure 8 Closed-form near-tip problem of an infinite plate with a through crack: (a) geometry and (b) scaled boundary model.

The relative error in the energy norm used in Example 1 is also employed in this problem to evaluate the computational accuracy of the different approaches. Figure 9 shows that the results are similar to the results obtained for Example 1. The higher-order elements and Fourier shape functions have better convergence than the other approaches, and the higher-order elements are the best overall. The EFG-SBM has better performance than the conventional SBFEM and MLPG-SBM.

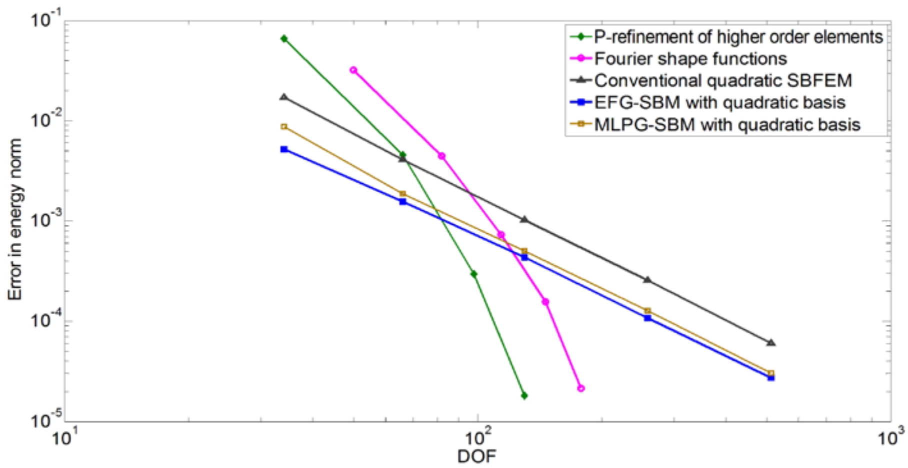


Figure 9 The variation of error in energy norm with degrees of freedom

Conclusions

In this paper various numerical discretisation approaches for the scaled boundary method are reviewed, and numerical examples are provided to compare the accuracy and convergence of these approaches. Compared with the conventional SBFEM with linear or quadratic elements, these approaches have better performance in accuracy and convergence. For the meshless scaled boundary method (EFG-SBM and MLPG-SBM), higher accuracy can be obtained than the conventional SBFEM using the same number of nodes and order of elements or basis functions. For the higher-order elements and Fourier shape functions, significant advantages in convergence rate can be obtained, with very high accuracy obtained as more degrees of freedom are introduced. However, at lower numbers of degrees of freedom, the meshless approaches often give greater accuracy.

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