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# A new scaled boundary finite element method using Fourier shape functions

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## Abstract

The scaled boundary finite element method (SBFEM) is a semi-analytical method, whose versatility, accuracy and efficiency are not only equal to, but potentially better than the finite element method and the boundary element method for certain problems. This paper investigates the possibility of using Fourier shape functions in the SBFEM to form the approximation in the circumferential direction. The shape functions effectively form a Fourier series expansion in the circumferential direction, and are augmented by additional linear shape functions. The proposed method is evaluated by solving elastostatic problems. The accuracy and convergence of the proposed method is demonstrated, and the performance is found to be better than using polynomial elements or using an element-free Galerkin approximation for the circumferential approximation.

**Keywords:** Scaled boundary method; Fourier shape functions; computational accuracy; stress singularities; unbounded domains

## Introduction

The scaled boundary method (SBM) is a semi-analytical method developed relatively recently by Wolf and Song (Wolf and Song, 1996). The method introduces a normalised radial coordinate system based on a scaling centre and a defining curve (usually taken as the boundary). The governing differential equations are weakened in the circumferential direction and then solved analytically in the normalised radial direction. The SBM combines the advantages of the Finite Element Method (FEM) and the Boundary Element Method (BEM), and, unlike the BEM, no fundamental solution is required. In addition, the SBM has been shown to be more efficient than the FEM for problems involving unbounded domains and for problems involving stress singularities or discontinuities (Deeks and Wolf, 2002). Effective applications of this method have been demonstrated in various problem domains, including fracture problems and foundation problems.

In the scaled boundary method, the discretisation approach used in the circumferential direction has significant influence on the accuracy of the resulting solutions (Deeks and Augarde, 2005). The most commonly used method for performing this circumferential discretisation is the finite element approach, leading to the method called the scaled boundary finite element method (SBFEM). Vu and Deeks (Vu and Deeks, 2006, 2008a, 2008b) investigated the use of higher-order polynomial shape functions in the SBFEM, and demonstrated the SBFEM converged significantly faster under p-refinement than under h-refinement. The development of meshless methods provided another approach to building circumferential approximations for the scaled boundary method. Deeks and Augarde (Deeks and Augarde, 2005) developed a Meshless Local Petrov-Galerkin method scaled boundary method (MLPG-SBM) and and He et al (He et al, 2012) developed an Element-free Galerkin scaled boundary method (EFG-SBM). This work showed that these two meshless scaled boundary methods gave a higher level of accuracy and rate of convergence than the conventional SBFEM using linear or quadratic elements, with the EFG-SBM performing slightly better than the MLPG-SBM.

In this paper, the possibility of using shape functions based on the terms of a Fourier series for the circumferential approximation of the SBFEM is investigated. Fourier interpolations containing trigonometric functions have been applied to both the finite element method (FEM) and the boundary element method (BEM). For example, Guan et al. (Guan et al, 2006) developed a Fourier series based FEM for the analysis of tube hydroforming, and showed that this Fourier shape function reduced the number of degrees of freedom required. Javaran and Khaji (Javaran, 2011; Khaji and Javaran, 2013) applied Fourier radial basis functions into the BEM, and showed that the resulting BEM is much more accurate than the BEM using classic Lagrange shape functions. Although the advantages of Fourier based FEM and BEM have been illustrated in previous work, to date there has been no work reported on the use of Fourier shape functions in the SBFEM.

A new Fourier-based scaled boundary method (F-SBM) is presented in this paper. A set of shape functions based on Fourier series expansion is derived, and augmented with linear shape functions. The new shape functions provide good approximation to both trigonometric and polynomial functions in the circumferential direction of the scaled boundary system. In the numerical example, the F-SBM is used to solve a two-dimensional elastostatic problem. The accuracy and convergence of F-SBM is compared with the conventional SBFEM using both linear and quadratic elements and with the EFG-SBM. Superior performance in terms of both accuracy and convergence is demonstrated.

## A Fourier shape function

This paper employs shape functions obtained from the well-known Fourier series. Based on the theory of the Fourier series, any continuous function $f(r)$ maybe represented by a series of trigonometric functions as

$$f(r) = a_0 + \sum_{n=1}^{\infty}\left( a_n \cos\left(\frac{n\pi}{L_{\max}} r\right) + b_n \sin\left(\frac{n\pi}{L_{\max}} r\right)\right) \tag{1}$$

where $a_0$, $a_n$, $b_n$ and $L_{\max}$ represent the Fourier series parameters.

Thus on the boundary at $\xi = 1$, the displacement can be approximated as

$$u_h(s) = a_0 + \sum_{j=1}^{m}\left( a_j \cos\left(\frac{j\pi}{L} s\right) + b_j \sin\left(\frac{j\pi}{L} s\right)\right) \tag{2}$$

where $s$ is the circumferential coordinate in scaled boundary element, $L$ is the length of the boundary and $m$ represents the order of Fourier series.

To preserve $C^0$ continuity between the edges or elements, linear polynomial functions terms are added into the standard Fourier approximation as

$$u_h(s) = \alpha_1 \frac{L-s}{L} + \beta_1 \frac{s}{L} + \sum_{j=1}^{m}\left( a_j \cos\left(\frac{j\pi}{L} s\right) + b_j \sin\left(\frac{j\pi}{L} s\right)\right) \tag{3}$$

where $\alpha_1$ and $\beta_1$ represent the values of the function at the end nodes of the element.

While it is possible to use the Fourier parameters as the unknown boundary parameters when solving the scaled boundary finite element equations, here the parameters in the Fourier expansion above are transferred to nodal values at equally spaced nodes along each element for ease of applying essential boundary conditions and enforcing $C^0$ continuity between elements. If $(2m+2)$ nodes are used, the nodal values vector $\{u\}$ can be related to the parameters in Equations (3) by

$$\{u\} = [T]\{\hat{u}\} \tag{4}$$

where $\{\hat{u}\} = \{\alpha_1 \quad a_1 \ \cdots \ a_m \quad b_1 \ \cdots \ b_m \quad \beta_1\}^T$, and $[T]$ is a transfer matrix assembled as

$$T_{ij} = \psi_j(S_i) \quad i, j = 1, 2 \ldots 2m+2 \tag{5}$$

where $S_i$ is the circumferential coordinate of the $i^{\text{th}}$ node, and the component functions of the Fourier expansion are

$$\psi_i(s) = \begin{cases} \dfrac{L-s}{L} & i = 1 \\ \cos\left(\dfrac{(i-1)\pi}{L} s\right) & 2 \le i \le m+1 \\ \sin\left(\dfrac{(i-1-m)\pi}{L} s\right) & m+2 \le i \le 2m+1 \\ \dfrac{s}{L} & i = 2m+2 \end{cases} \tag{6}$$

Inverting Equation (4), the parameters $\{\hat{u}\}$ in Equation (3) can be related to the nodal values $\{u_h\}$ by

$$\{\hat{u}\} = [T]^{-1}\{u\} \tag{7}$$

Thus the approximation for displacement can be rewritten as

$$u_h(s) = \{\psi\}[T]^{-1}\{u\} \tag{8}$$

The shape functions relating to the nodal displacements are hence

$$\{\varphi\} = \{\psi\}[T]^{-1} \tag{9}$$

and the shape function matrix for the scaled boundary method then becomes

$$[N(s)] = \begin{bmatrix} \varphi_1(s) & 0 & \ldots & \varphi_{2m+2}(s) & 0 \\ 0 & \varphi_1(s) & \ldots & 0 & \varphi_{2m+2}(s) \end{bmatrix} \tag{10}$$

Figure 3 plots these Fourier shape functions for $m = 2$, where 6 nodes are required.

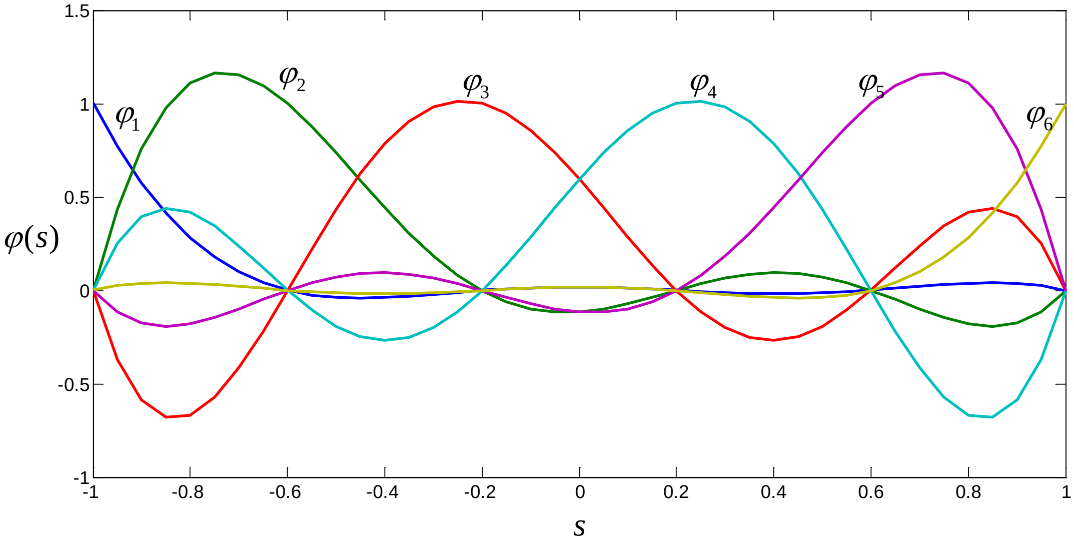


**Figure 1 The Fourier shape functions for order** $m = 2$

## Performance of the method

*An infinite plate with a through crack*

The example refers to the problem of determining the mode I stress intensity factor (SIF) $K_I$ for a through crack in an infinite plate, as illustrated in Figure 9. The applied stress $\sigma_0 = 1$. Due to the symmetry, one quarter of the problem is modelled, as shown in Figure 3, with the model consisting of a bounded domain, with the scaling centre at the crack tip (point E), and an unbounded domain, with the scaling centre at the middle of crack (point A). The nodes are introduced on the edges AB, BC and CD with uniform spacing, $ds$ . The problem has an exact solution, $K_I = \sigma_0 \sqrt{2\pi a}$ .

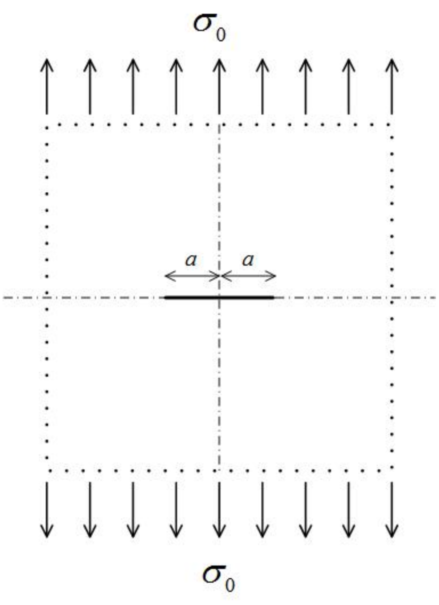


**Figure 2 Infinite plate with a through crack: geometry and loads**

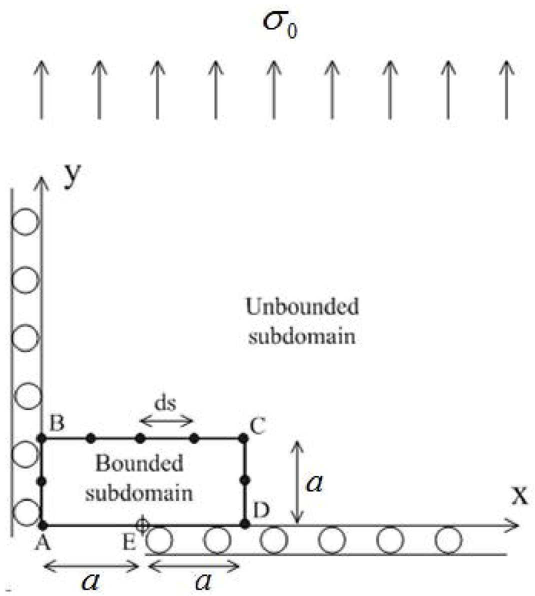


**Figure 3 Scaled boundary model of an infinite plate with a through crack.**

In Table 1 the F-SBM solutions are compared with the SBFEM with linear elements and the EFG-SBM with linear basis. The results show that the F-SBM achieves high accuracy for SIF, for example, a relative error as low as 0.0000555% can be obtained using 53 nodes. In comparison with SBFEM and EFG-SBM, it can be seen that F-SBM has higher accuracy when the same number of nodes are used.

**Table 1 The results of SIF using F-SBM**

| Number of nodes | **F-SBM** | Error% | **SBFEM (Linear)** | Error% | **EFG-SBM (Linear)** | Error% | **Exact Solutions** |
|---|---|---|---|---|---|---|---|
| 13 | 1.773047828 | 3.35e-2 | 1.765973947 | 3.65e-1 | 1.770648223 | 1.02e-1 | 1.772453851 |
| 21 | 1.771909603 | 3.07e-2 | 1.770015354 | 1.37e-1 | 1.772806367 | 1.99e-2 | |
| 28 | 1.772534634 | 4.56e-3 | 1.771193770 | 7.11e-2 | 1.772432519 | 1.20e-3 | |
| 37 | 1.772441208 | 7.13e-4 | 1.771691632 | 4.30e-2 | 1.772372556 | 4.58e-3 | |
| 45 | 1.772455534 | 9.49e-5 | 1.771443822 | 5.69e-2 | 1.772495119 | 2.33e-3 | |
| 53 | 1.772452866 | 5.55e-5 | 1.771456952 | 5.62e-2 | 1.772400534 | 3.01e-3 | |

## Conclusions

A new SBFEM using Fourier shape functions is presented in this paper. The shape functions are based on the Fourier series expansion and augmented with additional linear shape functions terms. By using a transfer matrix, the nodal values are related with Fourier parameters, and in this way the essential boundary conditions can be conveniently handled. In the numerical example, the new approach has been shown to yield higher accuracy and faster convergence in comparison with the SBFEM using linear or quadratic elements and the EFG-SBM using linear or quadratic basis.

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