Ultra-Accurate Isogeometric Structural Vibration Analysis with Novel Higher Order Mass Formulations

***Dongdong Wang, Xiwei Li, Wei Liu, and Hanjie Zhang**

Department of Civil Engineering, Xiamen University, Xiamen, Fujian 361005, China

*Corresponding author: ddwang@xmu.edu.cn

Abstract

An ultra-accurate isogeometric structural vibration analysis is presented. The key ingredient of the proposed methodology is the development of novel higher order mass matrices which are realized though a new two-step mass construction method. Firstly by using the standard consistent mass matrix a special reduced bandwidth mass matrix with equal order of accuracy is designed under the mass conservation constraint. A mixed mass matrix follows through a linear combination of the consistent mass matrix and the reduced bandwidth matrix. Subsequently the desired higher order mass matrix is then rationally deduced from the mixed mass matrix by optimizing the linear combination parameter in order to minimizing the frequency error. It turns out that for the semi-discrete free vibration analysis, the orders of accuracy associated with the proposed higher order mass matrices are two orders higher than those of their corresponding consistent mass formulations. Meanwhile, a detailed analysis of the full-discrete formulation with Newmark temporal integration demonstrates that the accuracy of the full-discrete frequency associated with the higher order mass matrices is superior compared with that of the standard consistent matrix matrices. The ultra-accurate performance of the proposed method is illustrated through several examples.

Keywords: isogeometric analysis, structural vibration, semi- and full-dsicretzation, frequency accuracy, higher order mass matrix.

Introduction

To seamlessly integrate the computer aided geometry design (CAGD) and the finite element analysis (FEA), Hughes et al. [1] proposed the isogeometric analysis where the CAGD data, i.e., the non-uniform rational B-splines (NURBS) and control points, is directly employed as the shape functions and geometry input for the finite element analysis. Thus exact geometry is preserved in the isogeometric analysis regardless of the model refinement. Meanwhile, high order smoothing convex approximation can be readily constructed for the NURBS basis functions, which makes isogeometric analysis ideal for the solution of problems with high order governing differential equations.

The excellent performance of isogeometric analysis has been demonstrated in many important problems [1-4], one of which is the structural vibration analysis. It has been shown by Cottrell et al. [5] and Reali [6] that the frequency spectra by isogeometric analysis are much more accurate than those by the typical higher order finite elements. Later Shojaee et al. [7] employed the isogeometric approach for free vibration analysis of thin plates. Thai et al. [8] investigated the static, free vibration, and buckling behaviors of laminated composite shear deformable plate with the isogeometric method. Very recently, Wang at al. [9] developed a set of novel higher order mass matrices for structural vibration analysis with ultra-accurate frequency accuracy. These higher order mass matrices are constructed by a two-step rational method. In the first stage, a reduced bandwidth mass matrix with the same order accuracy as the consistent mass matrix is developed, where the mass conservation is maintained. Then an optimal linear combination of the reduced bandwidth mass matrix and the consistent mass matrix yields the desired higher order mass matrix. An elevation of two orders of frequency accuracy is observed for the higher order mass matrix.

In this work the higher order mass matrix formulations for isogeometric analysis are first summarized, whose accuracy is demonstrated via classical free vibration examples. Thereafter a fully discrete formulation is introduced for the higher order mass isogeometric analysis to examine its discrete properties. The temporal discretization is completed by the widely used Newmark method. The full-discrete frequency is then derived with the aid of the semi-discrete frequency. Comparison between the full-discrete frequency and the continuum frequency is presented in detail. It turns out the higher order mass isogeometric analysis produces more favorable full-discrete frequency compared with the consistent mass formulation. Finally transient analysis results are also given to illustrate the proposed methodology.

Isogeometric Higher Order Mass Matrix

Isogeometric Basis Functions

The isogeometyric analysis often employs B-Spline and NURBS as the basis function for geometric description and finite element analysis. A set of *n p-th* order B-spline basis functions $N_{ap}(\xi)$'s are recursively defined as follows [1]:

$$N_{ap}(\xi) = N_{a(p-1)}(\xi)(\xi-\xi_a)/(\xi_{a+p}-\xi_a) + N_{(a+1)(p-1)}(\xi)(\xi_{a+p+1}-\xi)/(\xi_{a+p+1}-\xi_{a+1}) \text{ for } p \ge 1 \tag{1}$$

where in case of $p=0$, $N_{a0}(\xi)=1$ for $\xi_a \le \xi < \xi_{a+1}$ and otherwise $N_{a0}(\xi)=0$. ξ is the parametric coordinate, ξ_a is the *a-th* knot of the knot vector $\mathbf{k}_\xi = \{\xi_1 = 0, \cdots, \xi_a, \cdots, \xi_{n+p+1} = 1\}^T$. A NURBS basis function $R_a^p(\xi)$ is given by assigning a weight w_a to each B-spline basis function $N_{ap}(\xi)$:

$$R_a^p(\xi) = N_{ap}(\xi) w_a / \sum_{b=1}^{n} N_{bp}(\xi) w_b \tag{2}$$

Through tensor product operation, 2D NURBS basis function $R_{ab}^{pq}(\xi,\eta)$ takes the following form:

$$R_{ab}^{pq}(\xi,\eta) = N_{ap}(\xi) N_{bq}(\eta) w_{ab} / \sum_{c=1}^{n}\sum_{d=1}^{m} N_{cp}(\xi) N_{dq}(\eta) w_{cd} \tag{3}$$

where w_{ab} is the 2D weight for geometry description. $N_{bq}(\eta)$ is the *q-th* order basis function and m is the number of basis functions in the η direction, respectively.

Construction of Higher Order Mass Matrices

Here we consider the quadratic isogeometric approximation of an elastic rod with cross section area A and density ρ. In this case the element consistent mass matrix $\mathbf{M}^e$ and stiffness matrix $\mathbf{K}^e$ are given by [9]:

$$\mathbf{M}^{ec} = \frac{\rho A h}{120}\begin{bmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{bmatrix}, \quad \mathbf{K}^e = \frac{EA}{6h}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{4}$$

where h is the element length. Consider the classical semi-discrete vibration model problem:

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{0} \tag{5}$$

in which $\mathbf{d}$ contains the coefficients associated with the control points, the overhead dots represent time differentiation. It is noted that since NURBS basis functions are not interpolatory functions in general, $\mathbf{d}$ does not represent physical values of the control points and thus proper treatment of essential boundary conditions is required [10, 11]. Based on Eqs. (4) and (5), it is shown that the frequency error associated with the consistent mass matrix is [9]:

$$\omega^h / \omega \approx 1 + (kh)^4 / 1440 \tag{6}$$

where k is the wave number. ω and ω^h are the exact continuum frequency and the semi-discrete (spatially discrete) frequency. In [9], an equal order accurate reduced bandwidth mass matrix $\mathbf{M}^{er}$ can be postulated as follows:

$$\mathbf{M}^{er} = \frac{\rho h}{120}\begin{bmatrix} 7-\beta/2 & 13+r/2 & 0 \\ 13+\beta/2 & 54-r & 13+r/2 \\ 0 & 13+r/2 & 7-r/2 \end{bmatrix},\ \omega^h/\omega \approx 1-(4-r)(kh)^2/240+(18-r)(kh)^4/2880 \tag{7}$$

with r being an adjustable coefficient. Clearly selecting $r = 4$ gives us a 4$^{\text{th}}$ order accurate mass matrix that reduces the half-bandwidth of the consistent mass matrix by 1.

To establish a higher order mass matrix, we further consider the following mixed mass matrix $\mathbf{M}^{em}$ through linear combination of $\mathbf{M}^{ec}$ and $\mathbf{M}^{er}$:

$$\begin{cases} \mathbf{M}^{em} = (1-s)\mathbf{M}^{er} + s\mathbf{M}^{e} = \dfrac{\rho h}{120}\begin{bmatrix} 5+s & 15-2s & s \\ 15-2s & 50+4s & 15-2s \\ s & 15-2s & 5+s \end{bmatrix} \\ \omega^h/\omega \approx 1-(7-6s)(kh)^4/1440+(29-28s)(kh)^6/40320 \end{cases} \tag{8}$$

where s is a parameter. Thus we can choose $s = 7/6$ to achieve a 6$^{\text{th}}$ order accurate higher order mass matrix: $\mathbf{M}^{eho} = (7\mathbf{M}^{e} - \mathbf{M}^{er})/6$.

Though the tensor product formulation, the previous algorithm can be extended to construct multidimensional higher order mass matrix. In 2D case, we have the following quadratic higher order mass matrix [9]:

$$\mathbf{M}^{eho} = (13\mathbf{M}^{ec} - \mathbf{M}^{er})/12,\ \ \omega^h/\omega \approx 1+11(kh)^6/120960 \tag{9}$$

where $\mathbf{M}^{ec}$ and $\mathbf{M}^{er}$ are the 2D consistent and reduced bandwidth mass matrix whose explicit expressions can be found in [9].

Analysis of Fully Discrete Algorithm with Higher Order Mass Matrices

The analysis of the fully discrete algorithm can be completed for the following model problem through the standard modal reduction technique for Eq. (5):

$$\ddot{q} + (\omega^h)^2 q = 0 \tag{10}$$

where q is the generalized displacement. As for the temporal discretization, we consider the Newmark method. According to this method, the advancement of the variables such as displacement, velocity and acceleration at time t_n, i.e., $\{\mathbf{d}_n, \mathbf{v}_n, \mathbf{a}_n\}$, to their corresponding counterparts at t_{n+1}, say, $\{\mathbf{d}_{n+1}, \mathbf{v}_{n+1}, \mathbf{a}_{n+1}\}$, follows the following formula:

$$\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t \mathbf{v}_n + \frac{\Delta t^2}{2}\{(1-2\beta)\mathbf{a}_n + 2\beta \mathbf{a}_{n+1}\}, \ \mathbf{v}_{n+1} = \mathbf{v}_n + \Delta t\{(1-\gamma)\mathbf{a}_n + \gamma \mathbf{a}_{n+1}\} \tag{11}$$

where $\Delta t = t_{n+1} - t_n$, β and γ are parameters. Introducing Eq. (11) into Eq. (10) gives [12]:

$$\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n, \ \mathbf{y} = \{q, \dot{q}\}^T, \ \mathbf{A} = \begin{bmatrix} \dfrac{2-(1-2\beta)(\omega^h \Delta t)^2}{2[1+\beta(\omega^h\Delta t)^2]} & \dfrac{\Delta t}{1+\beta(\omega^h \Delta t)^2} \\ (\gamma-1)(\omega^h)^2 \Delta t & 1 \end{bmatrix} \tag{12}$$

with $\mathbf{A}$ being the amplification matrix. In the following discussion, $\gamma = 1/2$ is employed. The characteristic equation of $\mathbf{A}$ is:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - 2A_1\lambda + A_2 = 0, \ A_1 = 1 - (\omega^h \Delta t)^2 / [21 + \beta(\omega^h \Delta t)^2], \quad A_2 = 1 \tag{13}$$

where $\mathbf{I}$ is the 2 by 2 identity matrix, λ is the eigenvalue of $\mathbf{A}$ that has the following form:

$$\lambda = e^{i\alpha}, \ i = \sqrt{-1}, \ \alpha = \bar{\omega}^h \Delta t \tag{14}$$

with $\bar{\omega}^h$ being the full-discrete frequency that is different to semi-discrete frequency. The comparison of $\bar{\omega}^h$ and the exact continuum frequency ω is a useful index to measure the accuracy of the discrete algorithm. Substituting Eq. (14) into Eq. (13) leads to:

$$\sin^2(\bar{\omega}^h \Delta t / 2) = 1/[4\beta + (2/\omega^h \Delta t)^2] \tag{15}$$

For the 1D quadratic isogeometric higher order mass matrix we have [9]

$$\omega^h = \frac{c}{h}\sqrt{\frac{20[6-2\cos(2kh)-4\cos(kh)]}{2s\cos(2kh)+2(30-4s)\cos(kh)+60+6s}}, \quad s = 7/6 \tag{16}$$

where c is the wave speed. While in 2D case, ω^h is given by [9]:

$$\omega^h = \frac{c}{h}\sqrt{\frac{40[-\cos(4kh)-28\cos(3kh)-112\cos(2kh)-4\cos(kh)+145]}{\begin{bmatrix} s\cos(4kh)+52s\cos(3kh)+(900-92s)\cos(2kh) \\ +(3600-116s)\cos(kh)+2700+155s \end{bmatrix}}}, \quad s = 13/12 \tag{17}$$

Further substituting Eqs. (16) and (17) into (15) gives the respective fully discrete frequencies:
1D rod model problem:

$$\sin^2\frac{\bar{\omega}^h \Delta t}{2} = \frac{20\Delta t^2[6-2\cos(2kh)-4\cos(kh)]}{\begin{Bmatrix} 480\beta\Delta t^2 + 4h^2(60+6s) + (8sh^2 - 160\beta\Delta t^2)\cos(2kh) \\ +[8h^2(30-4s) - 320\beta\Delta t^2]\cos(kh) \end{Bmatrix}} \tag{18}$$

2D membrane model problem:

$$\left\{\sin^2\frac{\bar{\omega}^h \Delta t}{2}\right\}^{-1} = 4\beta + \frac{h^2\begin{bmatrix} s\cos(4kh)+52s\cos(3kh)+(900-92s)\cos(2kh) \\ +(3600-116s)\cos(kh)+2700+155s \end{bmatrix}}{10c^2(\Delta t)^2[145-\cos(4kh)-28\cos(3kh)-112\cos(2kh)-4\cos(kh)]} \tag{19}$$

Results and Discussions

Accuracy of Semi-discrete Frequency

First we consider a free vibration elastic rod problem, the geometry and material properties for the elastic rod are: length L=10, cross section area A=1, material density $\rho = 1$, and Young's modulus $E = 1$. Figure 1 list the fundamental frequency results for the vibrations of fixed-fixed, fix-free, free-fixed, free-free elastic rods using quadratic basis functions, where three types of mass formulations are compared, i.e., the consistent mass matrix "CM", the reduced bandwidth mass matrix "RBM", and the higher order mass matrix "HOM". The periodic basis functions are used to eliminate the boundary effect. The numerical results in Fig. 1 apparently demonstrate that the proposed higher order mass matrix has a 6th order of accuracy, while both the accuracy orders for the reduced bandwidth matrix and the standard consistent mass matrix are 4.

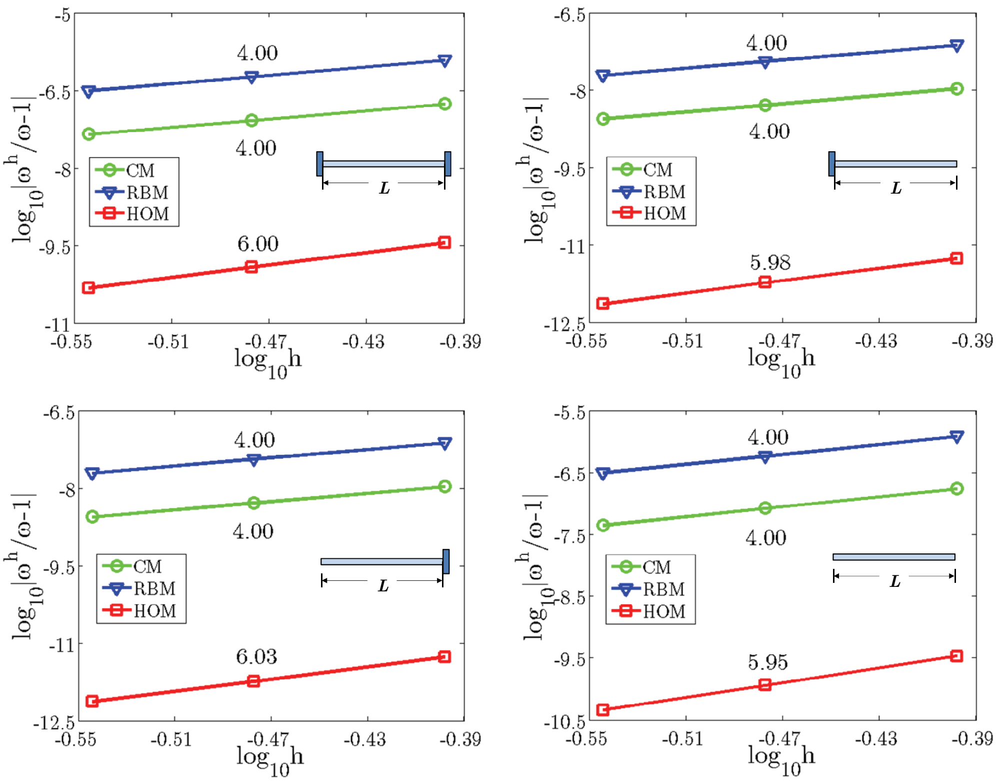

Figure 1. Comparison of ω^h and ω for 1D rod vibration problem

Accuracy of Full-discrete Frequency

The accuracy for the fully discrete algorithm with the proposed higher order mass matrix formulations is shown in Figs. 2-5. For convenience of presentation, the Courant number $C = c\Delta t / h$ is employed in the discussion. In Fig. 2 and 3, the 1D comparisons of the full-discrete and continuum frequency are plotted with respect to the element size and Courant number, four typical Newmark methods, i.e., central difference method ($\beta = 0$), Fox-Goodwin method ($\beta = 1/12$), linear acceleration method ($\beta = 1/6$) and average acceleration method ($\beta = 1/4$). The results reveal that in general the higher order mass formulation gives the most favorable full-discrete frequency accuracy. Similar conclusions are also observed for the frequency comparison for the 2D membrane model problem results as shown in Fig. 4. The transient analysis results of 2D fixed square membrane under the initial velocity $v_0 = \sin(\pi x)\sin(\pi y)$ in Fig. 5 once again demonstrate that the higher order mass formulation yields the superior solution accuracy, where unit geometric and material properties are adopted.

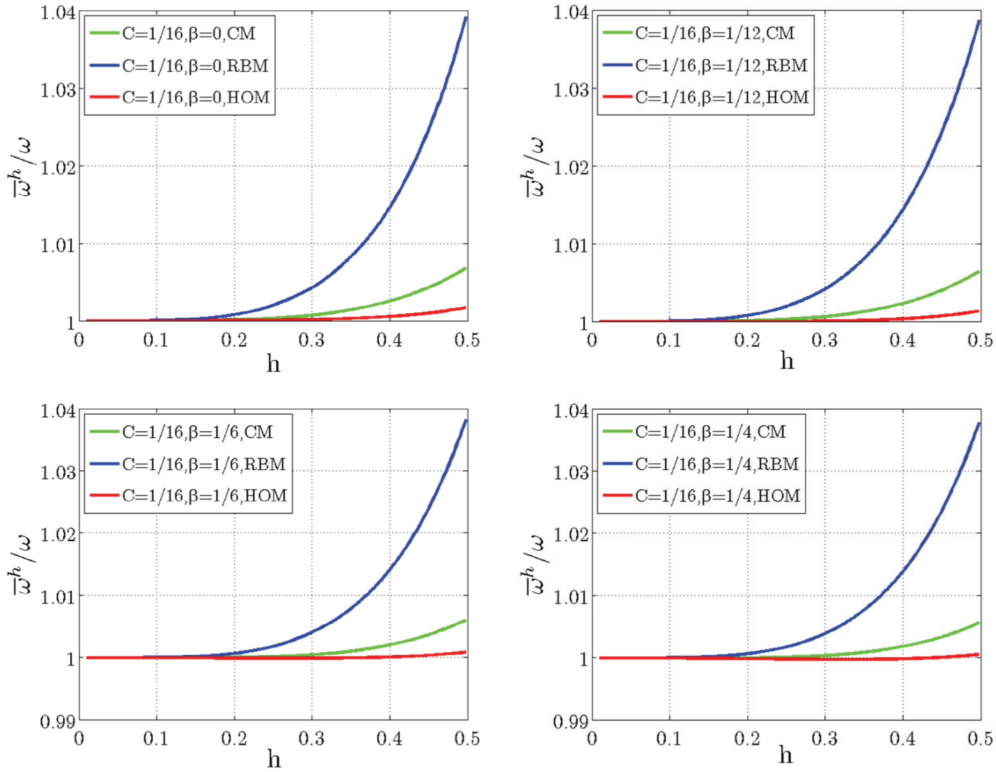

Figure 2. Comparison of $\overline{\omega}^h$ and ω with varying element size for 1D rod problem

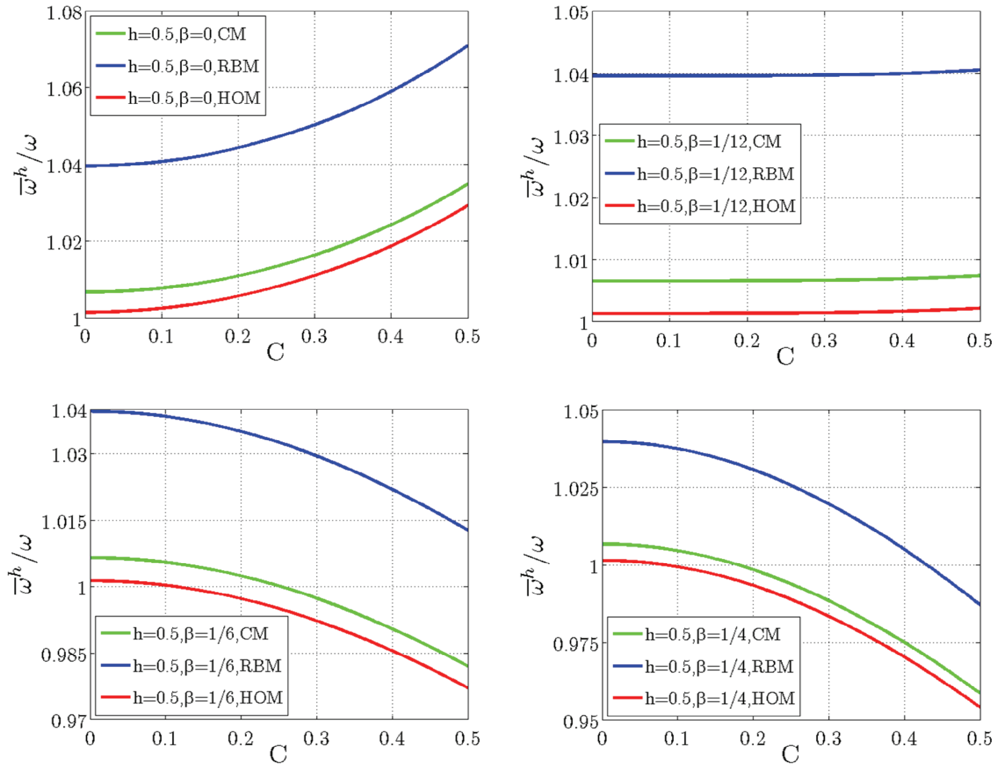

Figure 3. Comparison of $\overline{\omega}^h$ and ω with varying Courant number for 1D rod problem

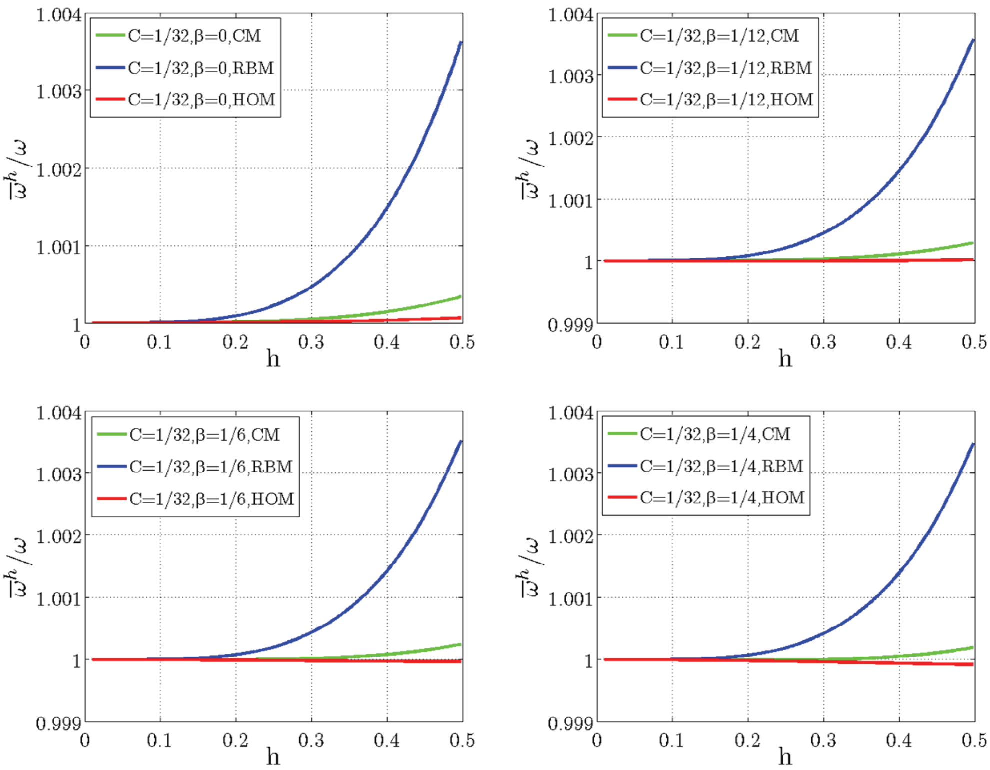

Figure 4. Comparison of $\bar{\omega}^h$ and ω with varying element size for 2D membrane problem

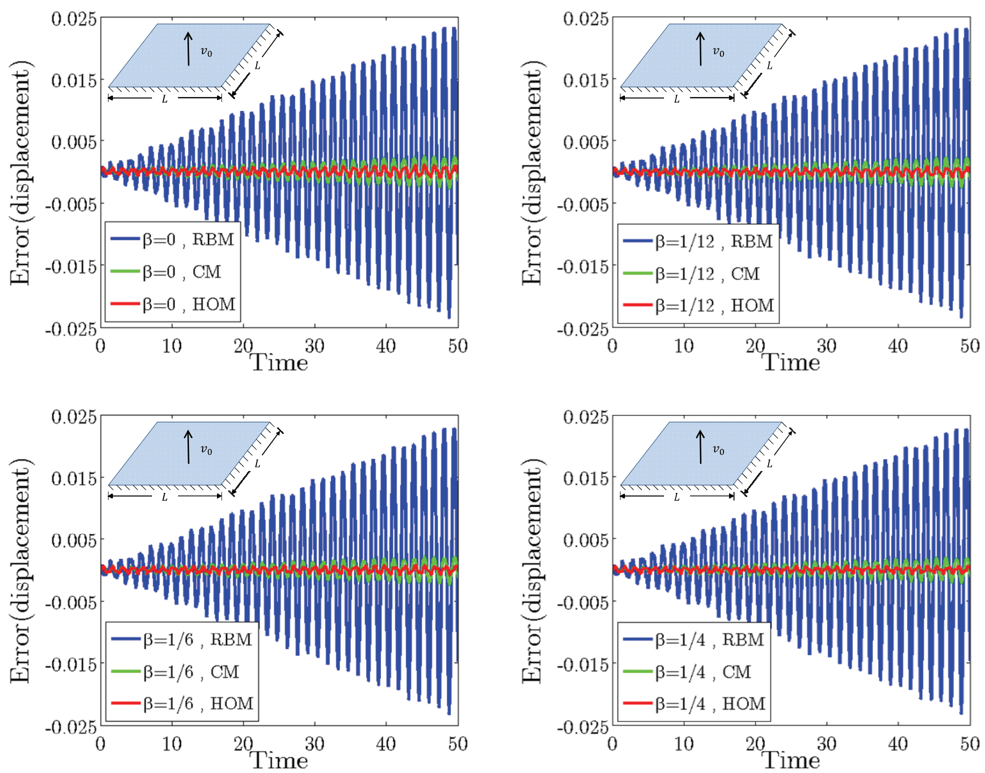

Figure 5. Comparison of the center deflection errors for a fixed square membrane under given initial velocity

Conclusions

An ultra-accurate isogeometric method was presented for structural vibration analysis. This method is featured by the novel higher order mass formulations. The higher order mass matrix was rationally formulated by optimally combining the consistent mass matrix and the so-called reduced bandwidth mass matrix that has an equal order of frequency accuracy with its consistent counterpart. For free vibration analysis, two orders of extra accuracy were gained by the higher order mass matrix. Furthermore, by introducing the Newmark time integration method, the accuracy of the fully discrete algorithm with the present higher order mass isogeometric approach is studied in detail. The full-discrete frequencies for four typical Newmark time integration methods, i.e., central difference method, Fox-Goodwin method, linear acceleration method and average acceleration method were compared with the continuum frequency with respect to the element size and the Courant number, respectively. Moreover, a 2D transient membrane example was also presented to investigate the dynamic response of the proposed method. All the numerical results universally demonstrated that the most favorable solution accuracy is attached with the proposed higher order mass isogeometric method.

Acknowledgements

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References

1. Hughes, T. J. R., Cottrell, J. A. and Bazilevs, Y. (2005), Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, *Computer Methods in Applied Mechanics and Engineering*, 194, pp. 4135-4195.
2. Verhoosel, C. V., Scott, M.A., and Hughes, T. J. R. (2011), An isogeometric analysis approach to gradient damage models, *International Journal for Numerical Methods in Engineering*, 86, pp. 115–134.
3. Lu, J. (2011), Isogeometric Contact analysis: geometric basis and formulation for frictionless contact, *Computer Methods in Applied Mechanics and Engineering*, 200, pp. 726-741.
4. Koo B., Yoon, M. and Cho, S. (2013), Isogeometric shape design sensitivity analysis using transformed basis functions for kronecker delta property, *Computer Methods in Applied Mechanics and Engineering*, 253, pp. 505-516.
5. Cottrell, J. A., Reali, A., Bazilevs, Y. and Hughes, T. J. R. (2006), Isogeometric analysis of structural vibrations, *Computer Methods in Applied Mechanics and Engineering*, 195, pp. 5257-5296.
6. Reali, A. (2006), An isogeometric analysis approach for the study of structural vibrations, *Journal of Earthquake Engineering*, 10, pp. 1–30.
7. Shojaee, S., Izadpanah, E., Valizadeh, N., and Kiendl, J. (2012), Free vibration analysis of thin plates by using a NURBS-based isogeometric approach, *Finite Elements in Analysis and Design*, 61 , pp. 23–34.
8. Thai, C. H., Nguyen-Xuan, H., Nguyen-Thanh, N., Le, T. H., Nguyen-Thoi, T., and Rabczuk, T. (2012), Static, free vibration, and buckling analysis of laminated composite Reissner-Mindlin plates using NURBS-based isogeometric approach, *International Journal for Numerical Methods in Engineering* , 91, pp. 571–603.
9. Wang, D., Liu, W. and Zhang H. (2013), Novel higher order mass matrices for isogeometric structural vibration analysis. *Computer Methods in Applied Mechanics and Engineering*, 260, pp. 92-108.
10. Wang, D. and Xuan, J. (2010), An improved NURBS-based isogeometric analysis with enhanced treatment of essential boundary conditions, *Computer Methods in Applied Mechanics and Engineering*, 199, pp. 2425-2436.
11. Wang, D., Xuan J., and Zhang, C. (2012), A three dimensional computational investigation on the influence of essential boundary condition imposition in NURBS isogeometric finite element analysis, *Chinese Journal of Computational Mechanics*, 29, pp. 31-37.
12. Hughes, T. J. R. (1983), Analysis of Transient Algorithms with Particular Reference to Stability Behavior, *Computational Methods for Transient Analysis*, eds. Belytschko, T. and Hughes, T. J. R., Amsterdam: North-Holland, pp. 67-155.