An optimal eighth-order scheme for multiple zeros of univariate functions

Ramandeep Behl [a], Fiza Zafar [b], Ali Saleh Alshomrani [c], †Moin-ud-Din Junjua [b] and Nusrat Yasmin [b]

[a] School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa

[b] Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan

[c] Department of Mathematics, King Abdulaziz University, Jeddah 21577, Saudi Arabia

†Corresponding author: moin_junjua@yahoo.com

Abstract

We present an optimal eighth-order scheme which will work for multiple zeros with multiplicity $(m \geq 1)$, for the first time. Earlier, the maximum convergence order of multi-point iterative schemes was six for multiple zeros in the available literature. So, the main contribution of this study is to present a new higher-order and as well as optimal scheme for multiple zeros for the first time. In addition, we present an extensive convergence analysis with the main theorem which confirms theoretically eighth-order convergence of the presented scheme. Moreover, we consider several real life problems which contain simple as well as multiple zeros in order to comparison with the existing robust iterative schemes. Finally, we conclude on the basis of obtained numerical results that the proposed iterative methods perform far better than the existing methods in terms of residual error, computational order of convergence and difference between the two consecutive iterations.

Keywords: Nonlinear equations, Kung-Traub conjecture, multiple zeros, efficiency index, optimal iterative methods.

Introduction

In the earlier years, it was very tough to construct a higher-order optimal multi-point scheme for multiple zeros of the involved function f with multiplicity $(m \geq 1)$. One of the main reason was the lengthy and complicated calculation which was quite tough or consume a lots of time to solve. Nowadays, with the advancement of digital computer, advanced computer arithmetics and symbolic computation, the construction of higher-order optimal multi-point methods become more vital and popular in this field. Because, the calculation of error equations of iterative methods and asymptotic error constant term for multiple zeros become easier now than the earlier time. However, still there is a need of hard work in order to construct higher-order optimal schemes.

Several scholars from worldwide like Li et al. [1] in (2009), Sharma and Sharma [2] and Li et al. [3] in (2010), Zhou et al. [4] in (2011), Sharifi et al. [5] in (2012), Soleymani et al. [6], Soleymani and Babajee [7], Liu and Zhou [8] and Zhou et al. [9] in (2013), Thukral [10] in (2014), Behl et al. [11] and Hueso et al. [12] in (2015) and Behl et al. [13] in (2016) have presented optimal fourth-order methods for multiple zeros in last two-three decades. In addition, Li et al. [3] (expect two of them are optimal) and Neta [14] presented non-optimal fourth-order iterative methods. Most of the above listed methods are the extension or modification of modified Newton's method (also known as Rall's method [20]) or Newton like method at the expense of additional functional evaluations or increase the substep of the original methods.

In the last two decades, many researchers from worldwide have tried to develop an optimal scheme whose convergence order should be greater than four (for multiple zeros with multiplicity $m \geq 1$ of univariate function). But, none of them have succeeded in this direction till date. However, some scholars have attained maximum sixth-order convergence in the case of multiple zeros which can be find in the available literature. There are only three multi-point iterative schemes with sixth-order convergence for multiple zeros till date, according to our best knowledge (which were proposed in the recent years). First one was proposed by Thukral [15] and other two were presented by Geum et al. [16, 17]. The details can be seen as follow:

In 2013, Thukral [15] presented a multi-point iterative method with sixth-order convergence, which is given by

$$\begin{aligned} y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - m\frac{f(x_n)}{f'(x_n)}\sum_{i=1}^{3} i\left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{i}{m}}, \\ x_{n+1} &= z_n - m\frac{f(x_n)}{f'(x_n)}\left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}\left[\sum_{i=1}^{3} i\left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{i}{m}}\right]^2. \end{aligned} \tag{1.1}$$

In 2015, Geum et al. [16], have given the following two-point sixth-order iterative scheme:

$$\begin{aligned} y_n &= x_n - m\cdot\frac{f(x_n)}{f'(x_n)}, m > 1, \\ x_{n+1} &= y_n - Q(p_n, s_n)\cdot\frac{f(y_n)}{f'(y_n)}, \end{aligned} \tag{1.2}$$

where, $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}, s_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$ and $Q: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic function in the neighborhood of origin $(0,0)$.

In 2016, Geum et al. [17], have again proposed a three-point iterative scheme with sixth-order convergence for multiple zeros. The proposed scheme was based on weight function approach, which can be seen in the following expression:

$$\begin{aligned} y_n &= x_n - m\cdot\frac{f(x_n)}{f'(x_n)}, m \geq 1, \\ w_n &= y_n - m\cdot G(p_n)\cdot\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= w_n - m\cdot K(p_n, t_n)\cdot\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1.3}$$

where, $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $t_n = \sqrt[m]{\frac{f(w_n)}{f(x_n)}}$. The weight functions $G: \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of 0 and $K: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic in a neighborhood of $(0,0)$.

All of the above three schemes (1.1), (1.2) and (1.3) require four functional evaluations in order to produce sixth-order convergence with the efficiency index $6^{\frac{1}{4}} = 1.5650$. So, none of them is optimal scheme according to the classical Kung-Traub's conjecture [18]. In addition, the above expression (1.2) has one more drawback that it does not work for simple zeros (i.e. $m = 1$). Moreover, there does not exist any optimal scheme whose convergence order is greater than four in the case of multiple zeros according to our best knowledge. So, we need optimal eighth-order schemes which will work for multiple zeros ($m > 1$) as well as for simple zeros ($m = 1$) because they have better efficiency index than fourth and sixth-order methods. Furthermore, these schemes also require a small number of iterations in order to obtain desired accuracy as compare to fourth and sixth-order methods.

Motivated and inspired by this, we present an optimal scheme with eighth-order convergence, which will work for multiple zeros with multiplicity $m \geq 1$, for the first time. The proposed scheme requires four functional evaluations in order to reach eighth-order convergence with the efficiency index $8^{\frac{1}{4}} = 1.6817$, which is higher than the efficiency index of any of the existing methods for multiple zeros in the available literature (also of the recent sixth-order schemes proposed by Thukral [15] and Geum et al. [16, 17]). The rest of the paper is organized as follows. Section 2 provides the methodology and convergence analysis for the proposed optimal eighth order scheme. In Section 3, some special cases of the new scheme are considered. Section 4 is devoted to numerical experiments and comparisons of different multiple zero finders using some real life problems. Finally, conclusions are given in Section 5.

Construction of optimal scheme with eighth-order convergence

This section is devoted to the main contribution of this study and convergence analysis of the proposed scheme with main theorem. Here, we consider the following proposed by Wang and Liu [19]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0,$$

$$z_n = y_n - \frac{f(x_n)}{f'(x_n)} G\left(\frac{f(y_n)}{f(x_n)}\right),$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[H\left(\frac{f(y_n)}{f(x_n)}\right) + W\left(\frac{f(z_n)}{f(y_n)}\right)V\left(\frac{f(y_n)}{f(x_n)}\right)\right], \tag{2.1}$$

Where $G, H, V, W: \mathbb{R} \to \mathbb{R}$ are the weight functions and sufficiently differentiable in the neighborhood of origin. The above scheme is an optimal eighth-order scheme for only simple zeros.

Now, we want to extend this scheme for multiple zeros with multiplicity $m \geq 1$. So, we will rewrite the above expression (2.1) in simpler form with some modifications in second and third substep, in the following way:

$$y_n = x_n - m \cdot \frac{f(x_n)}{f'(x_n)}, n \geq 0,$$

$$z_n = y_n - m \cdot u \cdot \frac{f(x_n)}{f'(x_n)} \frac{1+\beta u}{1+(\beta-2)u}, \beta \in \mathbb{R}$$

$$x_{n+1} = z_n - u \cdot v \cdot \frac{f(x_n)}{f'(x_n)}[\alpha_1 + (1+\alpha_2 v)P_f(u)], \tag{2.2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are two free disposable parameters and the weight function $P_f: \mathbb{C} \to \mathbb{C}$ is an analytic function in a neighborhood of (0) with $u = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}, v = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ are disposable parameters. It is worthy to note that we will obtain well known King's family of fourth-order iterative methods for $m=1$ with the help of first two substep. In addition, we can obtain an optimal eighth-order scheme for simple zeros as special case of Wang and Liu's scheme for $m=1$.

In the next Theorem 2.1, we demonstrate that the order of convergence of the proposed scheme will reach at optimal eight without using additional functional evaluations. It is interesting to observe that how P_f and disposable parameters $(\alpha_i, i=1,2)$ contributes their role in the construction of the desired eighth-order convergence (for the details please see the Theorem 2.1).

Theorem 2.1 *Let us consider $x=\xi$ (say) be a multiple zero with multiplicity $m \geq 1$ of the involved function f. In addition, we assume that $f: \mathbb{C} \to \mathbb{C}$ be an analytic function in the region enclosing a multiple zero ξ. The proposed scheme defined by (2.2) has an optimal eighth-order convergence, when it satisfies the following expressions*

$$\begin{aligned} &\alpha_1 = \frac{m}{2}, \alpha_2 = 2, P(0) = \frac{m}{2}, P'(0) = 2m, P''(0) = 2m(5-2\beta), \\ &P'''(0) = 12m(\beta^2 - 6\beta + 6), \end{aligned} \tag{2.3}$$

where $\beta \in \mathrm{R}$.

Proof. Let us assume that $e_n = x_n - \xi$ be the error at nth step. Now, expand $f(x_n)$ and $f'(x_n)$ about $x=\xi$ by the Taylor's series expansion (with the help of *Mathematica 11*), we have

$$f(x_n) = \frac{f^{(m)}(\xi)}{m!} e_n^m (1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)) \tag{2.4}$$

and

$$f'(x_n) = \frac{f^{(m)}(\xi)}{m!} e_n^{m-1}(m + c_1(m+1)e_n + c_2(m+2)e_n^2 + c_3(m+3)e_n^3 + c_4(m+4)e_n^4 + c_5(m+5)e_n^5$$

$$+ c_6(m+6)e_n^6 + c_7(m+7)e_n^7 + c_8(m+8)e_n^8 + O(e_n^9)), \qquad (2.5)$$

respectively, where $c_k = \frac{m!}{(m+k)!} \frac{f^{(m+k)}(\xi)}{f^{(m)}(\xi)}, k = 1,2,3,\ldots 8$.

By inserting the above expressions (2.4) and (2.5), in the first substep of (2.2), we will yield

$$y_n - \xi = \frac{c_1 e_n^2}{m} + \frac{(2mc_2 - (m+1)c_1^2)e_n^3}{m^2} + \sum_{k=0}^{4} A_k e_n^{k+4} + O(e_n^9), \qquad (2.6)$$

where $A_k = A_k(m, c_1, c_2, \ldots, c_8)$ are given in terms of $m, c_1, c_2, c_3, \ldots, c_8$ with explicitly written two coefficients $A_0 = \frac{1}{m^3}\{3c_3 m^2 + c_1^3(m+1)^2 - c_1 c_2 m(3m+4)\}$ and $A_1 = -\frac{1}{m^4}\{c_1^4(m+1)^3 - 2c_2 c_1^2 m(2m^2 + 5m + 3) + 2c_3 c_1 m^2(2m+3) + 2m^2(c_2^2(m+2) - 2c_4 m)\}$, etc.

With the help of Taylor's series expansion and expression (2.6), we have

$$f(y_n) = f^{(m)}(\xi) e_n^{2m} \left[\frac{(\frac{c_1}{m})^m}{m!} + \frac{(2mc_2 - (m+1)c_1^2)(\frac{c_1}{m})^m e_n}{m! c_1} + \sum_{k=0}^{6} \overline{A}_k e_n^{k+2} + O(e_n^9) \right]. \qquad (2.7)$$

By using the expressions (2.4) and (2.7), we get

$$u = \left(\frac{f(y_n)}{f(x_n)} \right)^{\frac{1}{m}} = \frac{c_1 e_n}{m} + \frac{(2mc_2 - (m+2)c_1^2)e_n^2}{m^2} + \tau_1 e_n^3 + \tau_2 e_n^4 + \tau_3 e_n^5 + O(e_n^6), \qquad (2.8)$$

where

$$\tau_1 = \frac{1}{2m^3}[c_1^3(2m^2 + 7m + 7) + 6c_3 m^2 - 2c_1 c_2 m(3m+7)],$$

$$\tau_2 = -\frac{1}{6m^4}[c_1^4(6m^3 + 29m^2 + 51m + 34) - 6c_2 c_1^2 m(4m^2 + 16m + 17) + 12c_1 c_3 m^2(2m+5)$$

$$+ 12m^2(c_2^2(m+3) - 2c_4 m)]$$

and

$$\tau_3 = \frac{1}{24m^5}[-24m^3(c_2c_3(5m+17)-5c_5m)+12c_3c_1^2m^2(10m^2+43m+49)+ 12c_1m^2\{c_2^2(10m^2+47m+53)-2c_4m(5m+13)\}-4c_2c_1^3m(30m^3+163m^2+306m+209)+ c_1^5(24m^4+146m^3+355m^2+418m+209)].$$

Now, insert the expressions (2.6) – (2.8) in the second substep of scheme (2.2), we obtain

$$z_n - \xi = \frac{(4\beta+m+1)c_1^3-2mc_1c_2}{2m^3}e_n^4 + \sum_{j=0}^{4} B_j e_n^{j+4} + O(e_n^9), \tag{2.9}$$

where $B_j = B_j(m, c_1, c_2, \ldots, c_8)$ are given in terms of $m, c_1, c_2, c_3, \ldots, c_8$ with explicitly written three coefficients $B_0 = -\frac{1}{6m^4}\{c_1^4(12\beta^2+36\beta+7m^2+12(3\beta+1)m+5)+12c_1c_3m^2+12c_2^2m^2-24c_2c_1^2m(3\beta+m+1)\}, B_1 = \frac{1}{24m^5}\{48\beta^3+144\beta^2+264\beta+46m^3+(288\beta+101)m^2+2m(96\beta^2+252\beta+37)+19\}$

and

$$B_2 = -\{12c_1^2c_3m^2(36\beta+13m+11)+(37-168c_2c_3m^3+4c_1^3c_2m(96\beta^2+252\beta+53m^2+ 18(14\beta+5)m)+12c_1m^2(c_2^2(48\beta+17m+19)-6c_4m)\},$$

etc.

Again with the help of above expression (2.9) and the Taylor's series expansion, we have

$$f(z_n) = f^{(m)}(\xi)e_n^{4m}\left[\frac{2^{-m}\left(\frac{(4\beta+m+1)c_1^3-2mc_1c_2}{m^3}\right)^m}{m!} - \frac{\left(2^{-m}\left(\frac{(4\beta+m+1)c_1^3-2mc_1c_2}{m^3}\right)^{m-1}\theta_0\right)}{3(m^3m!)}e_n + \sum_{j=0}^{7}\overline{B}_j e_n^{j+1} + O(e_n^9)\right], \tag{2.10}$$

where
$\theta_0 = \{12\beta^2+36\beta+7m^2+12m(3\beta+1)+5\}c_1^4+12m^2c_1c_3+12m^2c_2^2-24m(3\beta+m+1)c_1^2c_2.$

By using the above expressions (2.7) and (2.10), we further obtain

$$v = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}} = \frac{c_1^2(4\beta+m+1)-2mc_2}{2m^2}e_n^2 + \theta_1 e_n^3 + \theta_2 e_n^4 + \theta_3 e_n^5 + O(e_n^6), \tag{2.11}$$

Where $\theta_1 = -\frac{1}{3m^3}\{c_1^3(6\beta^2+12\beta+2m^2+3m(4\beta+1)+1)+6m^2c_3-6mc_1c_2(4\beta+$

$m+1)\}, \theta_2 ==$
$\frac{1}{24m^4}[-12c_2c_1^2m(24\beta^2+36\beta+6m^2+m(40\beta+7)-1)+24c_1c_3m^2(12\beta+3m+2)+12m^2(c_2^2(16\beta+3m+3)-6mc_4)+c_1^4\{48\beta^3+96\beta^2+72\beta+18m^3+(144\beta+25)m^2+6(24\beta^2+36\beta-1)m-13\}]$ and $\theta_3=-\frac{1}{60m^5}[120m^3(c_2c_3(12\beta+2m+1)-2mc_5)+60c_3c_1^2m^2(18\beta^2+24\beta+4m^2+28\beta m+3m-3)+60c_1m^2\{c_2^2(24\beta^2+24\beta+4m^2+32\beta m+3m-3)-2c_4m(8\beta+2m+1)\}-20c_2c_1^3m(48\beta^3+72\beta^2+36\beta+12m^3+(108\beta+11)m^2)+18(7\beta^2+8\beta-1)m-17)+c_1^5\{3(40\beta^4+80\beta^3+40\beta^2-21)+48m^4+10(48\beta+5)m^3+15(48\beta^2+56\beta-7)m^2+10(48\beta^3+72\beta^2+36\beta-17)m\}]$.

Since it is clear from the expression (2.8) that u is of order e_n. Therefore, we can expand weight function $P_f(u)$ in the neighborhood of origin by Taylor's series expansion up to third-order terms as follows:

$$P_f(u)=P(0)+P'(0)u+\frac{1}{2!}P''(0)u^2+\frac{1}{3!}P'''(0)u^3. \tag{2.12}$$

By using the expressions (2.4)–(2.12) in the last substep of proposed scheme (2.2), we have

$$e_{n+1}=\frac{c_1(c_1^2(4\beta+m+1)-2mc_2)(m-\alpha_1-\alpha_2P(0))}{2m^4}e_n^4+\sum_{i=1}^{4}E_ie_n^{i+4}+O(e_n^9), \tag{2.13}$$

where $E_i=E_i(m,\beta,\alpha_1,\alpha_2,P(0),P'(0),P''(0),P'''(0),c_1,c_2,\dots,c_8)$.

It is straightforward to say from the above expression (2.13) that we can easily obtain at least fifth-order convergence, when we will choose the following value of α_1

$$\alpha_1=m-P(0). \tag{2.14}$$

With the help of the above expression (2.14) and $E_1=0$, we obtain

$$2m-P'(0)=0, \tag{2.15}$$

which further yield

$$P'(0)=2m. \tag{2.16}$$

Again, inserting the above expressions (2.14) and (2.16) in $E_2=0$, we have

$$m-\alpha_2P(0)=0, -\alpha_2P(0)(4\beta+m+1)+m(m+11)-P''(0)=0, \tag{2.17}$$

the above two independent expressions, which further leads us

$$P(0)=\frac{m}{\alpha_2}, \quad P''(0)=2m(5-2\beta). \tag{2.18}$$

Now, by using the above expressions (2.14), (2.16), (2.18) and $E_3=0$, we obtain

$$\alpha_2 - 2 = 0,\quad 12m(\beta^2 - 2\beta + m + 7) - 6\alpha_2 m(4\beta + m + 1) - P'''(0) = 0, \tag{2.19}$$

which further yield

$$\alpha_2 = 2,\quad P'''(0) = 12m(\beta^2 - 6\beta + 6). \tag{2.20}$$

Finally, by substituting the above expressions (2.14), (2.16), (2.18) and (2.20) in the expression (2.13), we obtain the following optimal asymptotic error constant term

$$e_{n+1} = \frac{c_1((4\beta + m + 1)c_1^2 - 2mc_2)}{24m^7}[(347 - 24\beta^3 + 288\beta^2 - 492\beta + 7m^2 + 18m(2\beta + 1))c_1^4 + 12m^2 c_1 c_3$$

$$+12m^2 c_2^2 - 12m(6\beta + 2m + 3)c_1^2 c_2]e_n^8 + O(e_n^9). \tag{2.21}$$

The above asymptotic error constant (2.21) reveals that the proposed scheme (2.2) reaches at optimal eighth-order convergence by using only four functional evaluations (viz. $f(x_n), f'(x_n), f(y_n)$ and $f'(z_n)$) per iteration. This completes the proof.

Some special cases of weight function

In this section, we will discuss some special cases of our proposed scheme (2.2) by assigning different kind of weight functions P_f. In this regard, please see following cases, where we have mentioned some different kind of members of the proposed scheme:

Case A: Let us describe the following weight function directly from the proposed Theorem 2.1

$$P_f(u) = \frac{m}{2}\left(1 + 4u + (10 - 4\beta)u^2 + 4(\beta^2 - 6\beta + 6)u^3\right). \tag{3.1}$$

Thus, the corresponding optimal eighth-order iterative scheme is given by

$$\begin{aligned} y_n &= x_n - m\cdot\frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - m\cdot u\cdot\frac{f(x_n)}{f'(x_n)}\frac{1+\beta u}{1+(\beta-2)u}, \\ x_{n+1} &= z_n - \frac{m}{2}\cdot u\cdot v\cdot\frac{f(x_n)}{f'(x_n)}\left[1 + (2v+1)(4(\beta^2 - 6\beta + 6)u^3 + (10 - 4\beta)u^2 + 4u + 1)\right]. \end{aligned} \tag{3.2}$$

Case B: Now, we suggest rational weight function satisfying the conditions (2.3) as follows.

$$P_f(u) = -\frac{m((8\beta + 2)u^2 - (2\beta^2 - 4\beta - 8)u - 2\beta + 5)}{2(2(\beta^2 - 6\beta + 6)u + 2\beta - 5)}, \tag{3.3}$$

which further yields

$$
\begin{aligned}
y_n &= x_n - m\cdot\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - m\cdot u\cdot\frac{f(x_n)}{f'(x_n)}\frac{1+\beta u}{1+(\beta-2)u},\\
x_{n+1} &= z_n + \frac{m}{2}\cdot u\cdot v\cdot\frac{f(x_n)}{f'(x_n)}\left[1-\frac{(2v+1)(2u(2\beta-\beta^2+4\beta u+u+4)-2\beta+5)}{2\beta+2(\beta^2-6\beta+6)u-5}\right],
\end{aligned}
\tag{3.4}
$$

is a new optimal eighth-order scheme.

Case C: let us consider another rational weight function which satisfies the conditions of (2.3), is given by

$$
P_f(u) = \frac{m(2(\beta^2+2\beta+2)u+2\beta+3)}{2((8\beta+2)u^2+2(\beta^2-2\beta-4)u+2\beta+3)}. \tag{3.5}
$$

By using the above expression, we obtain the following optimal eighth-order scheme:

$$
\begin{aligned}
y_n &= x_n - m\cdot\frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - m\cdot u\cdot\frac{f(x_n)}{f'(x_n)}\frac{1+\beta u}{1+(\beta-2)u},\\
x_{n+1} &= z_n - m\cdot u\cdot v\cdot\frac{f(x_n)}{f'(x_n)}\left[\frac{(4\beta+1)u^2+2u(\beta^2+(\beta^2+2\beta+2)v-1)+(2\beta+3)(v+1)}{(8\beta+2)u^2+2(\beta^2-2\beta-4)u+2\beta+3}\right].
\end{aligned}
\tag{3.6}
$$

In the similar fashion, we can develop several new and interesting optimal schemes with eighth-order convergence for multiple zeros by just assigning different values to β or considering new weight functions which satisfy the conditions of Theorem 2.1.

Numerical experiments

This section is devoted to demonstrate the efficiency, effectiveness and convergence behavior of the presented scheme. In this regards, we consider some of the special cases of the proposed scheme namely, expression (3.2) $\left(\text{for } \beta = \frac{1}{2}\right)$, expression (3.4) $\left(\text{for } \beta = \frac{1}{3}\right)$ and expression (3.6) (for $\beta = 0$), denoted by $(M1), (M2)$ and $(M3)$, respectively. In addition, we choose a total number of five test problems for comparison: first one is eigen value problem; second one is Van der Waals equation which state the behavior of real gas; third one again is related to chemical reactor problem but for simple zeros; last two are standard test functions, which can be seen in the examples 4.1–4.5.

Now, we want to compare our methods with other existing robust methods of same order on the basis of difference between two consecutive iterations, computational order of convergence ρ and residual errors in the function. Unfortunately, there is no optimal eighth-order iterative methods for multiple zeros available in the literature in order to comparison. So, we have chosen sixth-order iterative methods for the comparison which is the highest-order till date for multiple zeros.

Therefore, we compare the proposed methods with the family of two-point sixth-order methods, which were presented by Guem et al. in [16], out of them we consider the following expression:

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, m > 1,\\
x_{n+1} &= y_n - m\left[1 + 2(m-1)(p_n - s_n) - 4p_n s_n + s_n^2\right]\cdot\frac{f(y_n)}{f'(y_n)},
\end{aligned}
\tag{4.1}
$$

called by $(GM1)$.

Finally, we compare them with another non-optimal scheme with sixth-order convergence based on weight function approach proposed by the same authors Guem et al. [17], out of them we chose the following expression:

$$
\begin{aligned}
y_n &= x_n - m\frac{f(x_n)}{f'(x_n)}, m \geq 1,\\
w_n &= x_n - m\left[1 + p_n + 2p_n^2\right]\cdot\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= x_n - m\left[1 + p_n + 2p_n^2 + (1 + 2p_n)t_n\right]\cdot\frac{f(x_n)}{f'(x_n)},
\end{aligned}
\tag{4.2}
$$

denoted by $(GM2)$.

In Tables 1–2, we display the number of iteration indexes (n), error in the consecutive iterations $|x_{n+1} - x_n|$, computational order of convergence (ρ) (we used the formula given by Cordero and Torregrosa [24] in order to calculate ρ) and absolute residual error of the corresponding function $(|f(x_n)|)$. We did our calculations with several number of significant digits (minimum 3000 significant digits) to minimize the round off error.

As we mentioned in the above paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the value of errors in the consecutive iterations $|x_{n+1} - x_n|$ and absolute residual errors in the function $|f(x_n)|$ up to 2 significant digits with exponent power in Tables 1–2. Moreover, computational order of convergence is up to 5 significant digits. Finally, we display the values of approximated zeros up to 30 significant digits in the examples.

For the computer programming, all computations have been performed using the programming package *Mathematica*11 with multiple precision arithmetic. Further, the meaning of $a(\pm b)$ is $a \times 10^{(\pm b)}$ in the following Tables 1–2.

Example 4.1. Eigen value problem:

One of the toughest and challenging task of linear algebra is concern with the eigen values of a large square matrix. Further, finding the zeros of characteristic equation of square matrix greater than 4 is another big challenge. So, we consider the following 9×9 matrix

$$A = \frac{1}{8}\begin{bmatrix} -12 & 0 & 0 & 19 & -19 & 76 & -19 & 18 & 437 \\ -64 & 24 & 0 & -24 & 24 & 64 & -8 & 32 & 376 \\ -16 & 0 & 24 & 4 & -4 & 16 & -4 & 8 & 92 \\ -40 & 0 & 0 & -10 & 50 & 40 & 2 & 20 & 242 \\ -4 & 0 & 0 & -1 & 41 & 4 & 1 & 2 & 25 \\ -40 & 0 & 0 & 18 & -18 & 104 & -18 & 20 & 462 \\ -84 & 0 & 0 & -29 & 29 & 84 & 21 & 42 & 501 \\ 16 & 0 & 0 & -4 & 4 & -16 & 4 & 16 & -92 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{bmatrix}.$$

The corresponding characteristic polynomial of the above matrix (A) is given as follows:

$$f_2(x) = x^9 - 29x^8 + 349x^7 - 2261x^6 + 8455x^5 - 17663x^4 + 15927x^3 + 6993x^2 - 24732x + 12960. \quad (4.3)$$

The above function has one multiple zero at $\xi = 3$ of multiplicity 4 with initial approximation $x_0 = 3.1$.

Example 4.2. Van der Waals equation of state

$$\left(P + \frac{a_1 n^2}{V^2}\right)(V - na_2) = nRT, \quad (4.4)$$

explains the behavior of a real gas by introducing in the ideal gas equations two parameters, a_1 and a_2, specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V.

$$PV^3 - (na_2P + nRT)V^2 + a_1n^2V - a_1a_2n^2 = 0. \quad (4.5)$$

Given the constants a_1 and a_2 of a particular gas, one can find values for n, P and T, such that this equation has a three real zeros. By using the particular values, we obtain the following nonlinear function

$$f_2(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675. \quad (4.6)$$

have three zeros and out of them one is a multiple zero $\xi = 1.75$ of multiplicity of order two and other one simple zero $\xi = 1.72$. However, our desired zero is $\xi = 1.75$. We considered initial guess $x_0 = 1.8$ for this problem.

Example 4.3. Fractional conversion in a chemical reactor:

Let us consider the following expression (for the details of this problem please see [26])

$$f_3(x) = \frac{x}{1-x} - 5\log\left[\frac{0.4(1-x)}{0.4-0.5x}\right] + 4.45977, \tag{4.7}$$

In the above expression x represents the fractional conversion of species A in a chemical reactor. Since, there will be no physical meaning of above fractional conversion if x is less than zero or greater than one. In this sense, x is bounded in the region $0 \le x \le 1$. In addition, our required zero to this problem is $\xi = 0.757396246253753879459641297929$. Moreover, it is interesting to note that the above expression will be undefined in the region $0.8 \le x \le 1$ which is very close to our desired zero. Furthermore, there are some other properties to this function which make the solution more difficult. The derivative of the above expression will be very close to zero in the region $0 \le x \le 0.5$ and there is an infeasible solution for $x = 1.098$. So, we consider the initial approximation $x_0 = 0.76$.

Example 4.4. Let us consider the following standard nonlinear test function from Behl et al. [13]

$$f_4(x) = \left(-\sqrt{1-x^2} + x + \cos\left(\frac{\pi x}{2}\right) + 1\right)^3 \tag{4.8}$$

The above function has a multiple zero at $\xi = -0.728584046444826716712333102423$ of multiplicity 3 with initial guess $x_0 = -0.69$

Example 4.5. We assume another standard test problem from Petkovíc et al. [21], which is defined by

$$f_5(x) = -\frac{x^4}{12} + \frac{x^2}{2} + x + e^x(x-3) + \sin(x) + 3, \tag{4.9}$$

This function f_5 has multiple zero at $\xi = 0$ of multiplicity 3. We will start with the initial approximation $x_0 = 0.6$ for this problem.

Table 1. Difference between two consecutive iterations (i.e. $|x_{n+1} - x_n|$) of different iteration functions.

$f_i(x)$	n	$GM1$	$GM2$	$M1$	$M2$	$M3$
$f_1(x)$	1	5.2(−3)	1.3(−2)	2.8(−2)	2.8(−2)	2.9(−2)
	2	2.5(−11)	2.4(−13)	4.0(−15)	2.8(−15)	1.9(−15)
	3	5.0(−33)	9.8(−78)	8.2(−118)	3.1(−119)	9.6(−121)
	ρ	2.9610	5.9962	7.9925	7.9939	7.9948

$f_2(x)$	1	8.1(–4)	1.1(–3)	2.6(–4)	2.1(–4)	1.6(–4)
	2	4.5(–12)	4.7(–11)	3.6(–19)	3.0(–20)	1.3(–21)
	3	1.6(–61)	5.0(–55)	6.1(–138)	5.2(–147)	3.1(–158)
	ρ	5.9908	5.9840	7.9977	7.9987	8.0512
$f_3(x)$	1	*	5.4(–9)	5.1(–12)	1.1(–13)	2.2(–14)
	2	*	4.6(–43)	1.2(–81)	6.4(–97)	5.2(–103)
	3	*	1.9(–247)	1.5(–638)	1.1(–762)	4.9(–812)
	ρ	**	6.0000	8.0000	8.0000	8.0512
$f_4(x)$	1	1.6(–8)	2.7(–8)	1.0(–10)	8.3(–11)	6.5(–11)
	2	2.5(–46)	9.2(–45)	7.0(–79)	1.0(–79)	9.7(–81)
	3	3.5(–273)	1.3(–263)	3.5(–624)	5.6(–631)	2.6(–639)
	ρ	6.0000	6.0000	8.0000	8.0000	8.0000
$f_5(x)$	1	5.8(–8)	2.0(–6)	4.4(–8)	4.3(–8)	4.3(–8)
	2	1.2(–47)	5.2(–38)	5.6(–64)	4.6(–64)	3.5(–64)
	3	9.4(–286)	1.4(–227)	4.2(–511)	6.9(–512)	6.0(–513)
	ρ	6.0000	6.0000	8.0000	8.0000	8.0000

(* means method is not working for simple zero ($m = 1$), ** means COC (ρ) can't be calculated for this method.)

Table 2. Comparison based on residual error (i.e. $|f(x_n)|$) of different iteration

functions.

$f_i(x)$	n	$GM1$	$GM2$	$M1$	$M2$	$M3$
$f_1(x)$	1	5.8(–12)	2.3(–6)	5.4(–5)	5.4(–5)	5.4(–5)
	2	2.9(–41)	2.5(–49)	2.1(–56)	5.0(–57)	1.1(–57)
	3	4.9(–128)	7.5(–307)	3.7(–467)	7.4(–473)	6.8(–479)
$f_2(x)$	1	2.0(–8)	3.4(–8)	2.0(–9)	1.4(–9)	7.9(–10)
	2	6.0(–25)	6.6(–23)	4.0(–39)	2.7(–41)	5.4(–44)
	3	7.5(–124)	7.5(–111)	1.1(–276)	8.1(–295)	2.9(–317)
$f_3(x)$	1		4.3(–7)	4.1(–10)	8.5(–12)	1.8(–12)
	2		3.7(–41)	9.9(–80)	5.1(–95)	4.2(–101)
	3		1.5(–245)	1.2(–636)	8.4(–761)	3.9(–810)
$f_4(x)$	1	1.1(–23)	5.2(–23)	2.7(–30)	1.5(–30)	6.9(–31)
	2	4.2(–127)	2.0(–132)	8.6(–235)	2.7(–237)	2.4(–240)
	3	1.1(–817)	6.0(–789)	1.1(–1870)	4.4(–1891)	4.7(–1916)
$f_5(x)$	1	3.3(–23)	1.4(–18)	1.4(–23)	1.4(–23)	1.3(–23)
	2	2.9(–142)	2.3(–113)	2.9(–191)	1.6(–191)	6.9(–192)
	3	1.4(–856)	4.7(–682)	1.2(–1532)	5.4(–1535)	3.6(–1538)

Conclusions

In this paper, we present an optimal eighth-order iterative scheme for finding multiple zeros of the involved function f with multiplicity $m \geq 1$, for the first time. An extensive convergence analysis is done which confirms theoretically eighth-order convergence of the proposed scheme. In addition, the proposed scheme is optimal in the sense of classical Kung-Traub conjecture. The beauty of the proposed methods is that they have not only smaller errors difference between two consecutive iterations and minimum residual errors corresponding to the considered test functions f_i. But, they also demonstrate the stable computational order of convergence as compared to the other listed methods. Further, the computational efficiency index of the proposed schemes is $E = \sqrt[4]{8} \approx 1.682$ which is better than the efficiency index of classical Newton's method $E = \sqrt[2]{2} \approx 1.414$ and also the schemes proposed by Thukral [15] and Guem et al. [16, 17], $E = \sqrt[4]{6} \approx 1.565$. Moreover, we can obtain several new optimal and interesting iterative methods of order eight by considering different types of weight functions and assigning different values to disposable parameter β. Finally, on accounts of the numerical results obtained, it can be concluded that our proposed methods are highly efficient and perform better than the existing methods for multiple zeros.

References

[1] Li, S., Liao, X. and Cheng L. (2009) A new fourth-order iterative method for finding multiple roots of nonlinear equations, *Applied Mathematics and Computation* **215** 1288–1292.
[2] Sharma, J. R., Sharma, R. (2010) Modified Jarratt method for computing multiple roots, *Applied Mathematics and Computation* **217** 878–881.
[3] S.G. Li, L.Z. Cheng, B. Neta, Some fourth-order nonlinear solvers with closed formulae for multiple roots, Comput. Math. Appl. 59 (2010) 126–135.
[4] X. Zhou, X. Chen, Y. Song, Constructing higher-order methods for obtaining the muliplte roots of nonlinear equations, J. Comput. Math. Appl. 235 (2011) 4199–4206.
[5] M. Sharifi, D.K.R. Babajee, F. Soleymani, Finding the solution of nonlinear equations by a class of optimal methods, Comput. Math. Appl. 63 (2012) 764–774.
[6] F. Soleymani, D.K.R. Babajee, T. Lofti, On a numerical technique forfinding multiple zeros and its dynamic, J. Egypt. Math. Soc. 21 (2013) 346–353.
[7] F. Soleymani, D.K.R. Babajee, Computing multiple zeros using a class of quartically convergent methods, Alex. Eng. J. 52 (2013) 531–541.
[8] B. Liu, X. Zhou, A new family of fourth-order methods for multiple roots of nonlinear equations, Non. Anal. Model. Cont. 18(2) (2013) 143–152.
[9] X. Zhou, X. Chen, Y. Song, Families of third and fourth order methods for multiple roots of nonlinear equations, Appl. Math. Comput. 219 (2013) 6030–6038.
[10] R. Thukral, A new family of fourth-order iterative methods for solving nonlinear equations with multiple roots, J. Numer. Math. Stoch. 6 (1) (2014) 37–44.
[11] R. Behl, A. Cordero, S.S. Motsa, J.R. Torregrosa, On developing fourth-order optimal families of methods for multiple roots and their dynamics, Appl. Math. Comput. 265(15) (2015) 520–532.
[12] J.L. Hueso, E. Martínez, C. Teruel, Determination of multiple roots of nonlinear equations and applications, J. Math. Chem. 53 (2015) 880-892.
[13] R. Behl, A. Cordero, S.S. Motsa, J.R. Torregrosa, V. Kanwar, An optimal fourth-order family of methods for multiple roots and its dynamics, Numer. Algor. 71 (4) (2016) 775–796.
[14] B. Neta, Extension of Murakami's high-order non-linear solver to multiple roots, Int. J. Comput. Math. 87(5) (2010) 1023–1031.
[15] R. Thukral, Introduction to higher-order iterative methods for finding multiple roots of nonlinear equations, J. Math. 2013 (2013) Article ID 404635, 3 pages http://dx.doi.org/10.1155/2013/404635.
[16] Y.H. Geum, Y.I. Kim, B. Neta, A class of two-point sixth-order multiple-zero finders of modified double-Newton type and their dynamics, Appl. Math. Comput. 270 (2015) 387–400 .
[17] Y.H. Geum, Y.I. Kim, B. Neta, A sixth-order family of three-point modified Newton-like multiple-root finders and the dynamics behind their extraneous fixed points, Appl. Math. Comput. 283 (2016) 120–140.
[18] H.T. Kung, J.F. Traub, Optimal order of one-point and multipoint iteration, J. Assoc. Comput. Mach. 21 (1974) 643–651.
[19] X. Wang, L. Liu, New eighth-order iterative methods for solving nonlinear equations, J. Comput. Appl. Math. 234 (2010) 1611–1620.
[20] J.F. Traub, Iterative methods for the solution of equations, Prentice-Hall, Englewood Cliffs, 1964.

[21] M.S. Petkovi c′ , B. Neta, L.D. Petkovi c′ , J. Džuni c′ , Multipoint methods for solving nonlinear equations, Academic Press, (2013)
[22] G.V. Balaji, J.D. Seader, Application of interval Newton's method to chemical engineering problems, Rel. Comput. 1 (3) (1995) 215–223.
[23] M. Shacham, An improved memory method for the solution of a nonlinear equation, Chem. Eng. Sci. 44 (7) (1989) 1495–1501.
[24] A. Cordero, J.R.Torregrosa, Variants of Newton's method using fifth-order quadrature formulas, Appl. Math. Comput. 190 (1) (2007) 686–698 .
[25] A. M. Ostrowski, Solution of equations and systems of equations, Academic Press, New York (1960).
[26] M. Shacham, Numerical solution of constrained nonlinear algebraic equations, Int. J. Numer. Method Eng. 23 (1986), 1455–1481.