Using the enriched radial basis function in solving the singular sudden expansion incompressible fluid flow problem

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Abstract

When incompressible viscous fluid flows through a channel with a sudden expansion with an re-entrant angle shown in Figure 1, singularity occurs exactly at the angle.

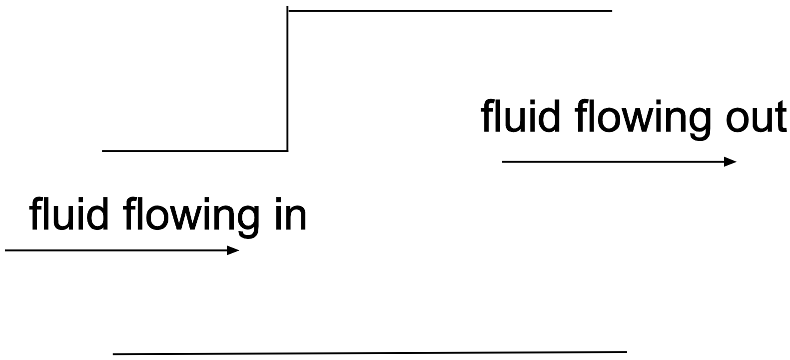

Figure 1 A channel with a sudden expansion

The governing equation is the standard Navier-Stokes equation. One of the form is expressed below which uses the string functionψ:

$$\rho\left(\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial x^2\partial y}\right)-\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y^3}+\frac{\partial^3\psi}{\partial x^2\partial y}\right)\right)=$$

$$\mu\left(\frac{\partial^4\psi}{\partial x^4}+\frac{2\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}\right)$$

However, the behavior of the stream function ψ near the re-entrant angle can be simplified as:

$$\frac{\partial^4\psi}{\partial x^4}+\frac{2\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0$$

To find the solution of ψ near the re-entrant angle, we would use polar coordinates and the equation becomes:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\psi=0$$

To solve this equation, we assume that:

$$\psi = r^{\lambda} f(\theta)$$

Put this into the governing equation:

$$r^{\lambda-4}(f'''' + ((\lambda-2)^2 + \lambda^2)f'' + (\lambda-2)^2\lambda^2 f) = 0$$

Therefore, the characteristic equation is:

$$x^4 + ((\lambda-2)^2 + \lambda^2)x^2 + (\lambda-2)^2\lambda^2 = 0$$

which have roots:

$$x = \pm\lambda i, \pm(\lambda-2)i$$

Therefore

$$f(\theta) = A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta$$

The boundary condition is that:

$$f(\pm\alpha) = f'(\pm\alpha) = 0$$

So, we have:

$$\sin 2(\lambda-1)\alpha = (\lambda-1)\sin 2\alpha$$
$$\sin 2(\lambda-1)\alpha = -(\lambda-1)\sin 2\alpha$$

These two equations give the value of λ that satisfy the boundary conditions.

For $\alpha = \frac{3}{4}\pi$, the values of λ that satisfy either of the two equations are listed below:

i	$\sin 2(\lambda-1)\alpha = (\lambda-1)\sin 2\alpha$	$\sin 2(\lambda-1)\alpha = -(\lambda-1)\sin 2\alpha$
1	1.54448	1.90853
2	2.62926±0.23125i	3.30133±0.31584i
3	3.97184±0.37393i	4.64142±0.41879i
4	5.31038±0.45549i	5.97890±0.48663i
5	6.64711±0.51368i	7.31508±0.53763i
6	7.98287±0.55911i	8.65051±0.57859i
7	9.31803±0.59642i	9.98546±0.61285i

The first roots of the two equations 1.54448 and 1.90853 are the main source of singularity at the reentrant angle as after differentiating twice, the stress can be found to contain the terms $r^{-0.45552}$ and $r^{-0.09147}$ which approach infinity when r is approaching 0.

In this paper, we are using the radial basis function method to model the sudden expansion problem. The method approximates a function f(x) by the following equation:

$$f(x) \approx \widetilde{f(x)} = \sum_{r=1}^{n} a_i \phi(\|\mathbf{x} - \mathbf{x}_i\|)$$

where $\mathbf{x}_i$'s are some nodal points in the studied domain and ϕ is a radial basis function. In order to have a better approximation of the singular solution, we add several terms from the series above to the collocation of the approximation of the approximation:

$$f(x) \approx \widetilde{f(x)} = \sum_{r=1}^{n} a_i \phi(\|\mathbf{x} - \mathbf{x}_i\|) + \sum_{i=1}^{m} b_i\, r^{\xi_i} h_i(\theta)$$

where (r,θ) are the polar coordinates near the re-entrant angle. We found that the accuracy first increase with the addition of a few terms from the series but then drop again as more terms are added. This is because the series is not convergent and therefore adding too many terms actually decrease the accuracy.