Axial Green's function Methods on Free Grids

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Abstract

We are going to talk about axial Green's function methods (AGMs) on free grids called axial lines. These are novel approaches in numerical computations. AGMs that we have developed for elliptic boundary value problems [3] and the steady Stokes flows [2] in complicated geometry use axial lines for discretization. These axial lines are parallel to axes and there is no restriction on their distribution. The salient feature of the methods is that not only one-dimensional Green's function for the axially split differential operators is sufficient to solve the multi-dimensional problems but also the free grids are available. In this talk, short introduction to AGMs is presented and then we show that the localization [1] of axial lines enables us to enforce Neumann boundary condition, and refinement of axial lines on separated regions are readily available as well.

Keywords: Axial Green's function, Free grids, Axial lines, One-dimensional Green's function, Multi-dimensional problem, Boundary condition, Refinement.

Introduction

By the axial Green's function, we mean that it is one-dimensional Green's function of an ordinary differential operator defined on lines parallel to axis, belonging to the multi-dimensional domain. In general, the finite difference method uses this kind of lines, called the grids, but the admissible grids in this method are so restrictive that the method cannot work unless the domain is simple or the grids are gradually changing in space. The axial Green's function methods(AGM) we have developed work fine in arbitrary domains without deterioration of accuracy, and furthermore they do even in randomly spacing axial lines.

The use of Green's function take place in the boundary element(BEM) method, which can reduce the dimension of the problem by discretizing the boundary of the domain. This is possible only when finding the fundamental solution or Green's function of the multi-dimensional differential operator, called partial differential operator. The BEM has been successful in Laplace operator, Lame operator in linear elasticity, Stokes operator in fluid mechanics, Helmholtz operator, and so on. However, if the material coefficients are functions of space variable, then the BEM suffers from finding the multi-dimensional Green's function in the domain or even a fundamental solution in entire space.

The advantages of AGMs are obvious in two points: (1) Arbitrarily distributed axial lines are available, which is inconvenient in FDMs, and (2) It is much easier than BEMs to find one-dimensional Green's functions. Based on these facts, we are able to implant these advantages to the refinements of axial lines in some regions of interest. The refined regions can be independently handled by using the representation formula for the solution in terms of axial Green's functions.

Axial Green's function method

For the sake of simplicity, we consider the Poisson problem in 2-dimensional domain Ω as an example:

$$-\Delta u = f, \quad \text{in } \Omega, \tag{1}$$

$$u = u^{\partial\Omega}, \quad \text{on } \partial\Omega. \tag{2}$$

Our interest is laid on the point that this multi-dimensional problem can be reformulated by one-dimensional problems. First of all, decomposing the multi-dimensional operator $-\Delta$ as two parts by introducing a new variable $\phi(x, y)$ as follows:

$$-u_{xx} = \phi, \quad \text{in } \Omega, \tag{3}$$

$$-u_{yy} = f - \phi, \quad \text{in } \Omega, . \tag{4}$$

From the first equation in (3), we find one-dimensional Green's functions to represent the

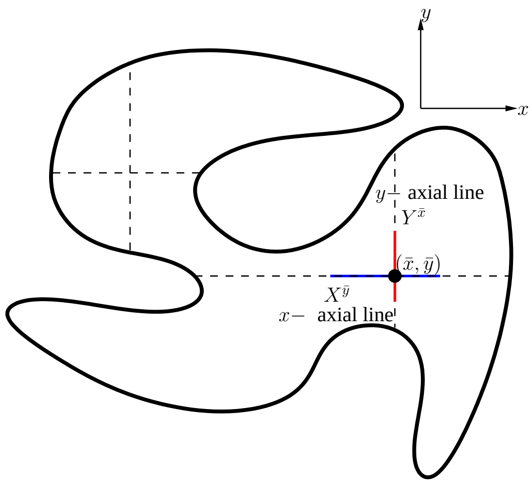

Figure 1: Axial lines for AGMs

solution $u(x, y)$ on $x-$axial line $X^{\bar{y}}$ and $y-$axial line $Y^{\bar{x}}$ associated with a given cross point $(\bar{x}, \bar{y}) \in \Omega$ as shown in Fig. 1:

$$u(\xi, \bar{x}) = \int_{X^{\bar{y}}} G(x, \xi; X^{\bar{y}})\phi(x, \bar{y})\, dx + u(x_-, \bar{y})B_-^X(\xi) + u(x_+, \bar{y})B_+^X(\xi),\ (\xi, \bar{y}) \in X^{\bar{y}}, \tag{5}$$

$$u(\bar{x}, \eta) = \int_{Y^{\bar{x}}} G(y, \eta; Y^{\bar{x}})(f - \phi)(\bar{x}, y)\, dy + u(\bar{x}, y_-)B_-^Y(\eta) + u(\bar{x}, y_+)B_+^Y(\eta),\ (\bar{x}, \eta) \in Y^{\bar{x}}. \tag{6}$$

In this case, these representations can be unified in the following form:

$$u(\tau) = \int_{t_-}^{t_+} G(t, \tau)g(t)\, dt + u(t_-)B_-(\tau) + u(t_+)B_+(\tau), \quad \tau \in (t_-, t_+), \tag{7}$$

where $G(t, \tau)$ is the corresponding one-dimensional Green's function and $B^{\pm}(\tau)$ is the function related to the boundary values $u(t_{\pm})$. Instead of directly attacking the multi-dimensional problem in (1) with boundary condition (2), we pay attention to the equations of integral form in (3) and (4). That is, after discretizing these integral equations for the unknown ϕ and u, we can solve the resultant system of equations well using GMRES.

In an analogous way, AGM can be applied to more general problems, for instance, general elliptic problem with function coefficient, the Stokes flow, and the convection-diffusion problem with variable coefficients, etc. For more effective computations, we need grid refinements in

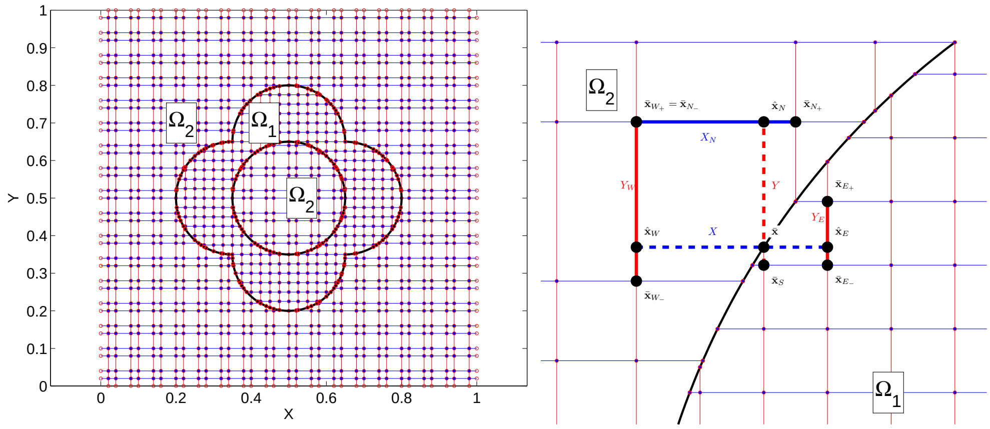

Figure 2: (left) Split domains Ω_1 and Ω_2 and (right) Interfacial configutration between split domains for refinement

different subdomains of interest. Let us consider the following problem:

$$-\nabla \cdot (\epsilon \nabla u) + \mathbf{U} \cdot \nabla \mathbf{u} = f, \quad \text{in } \Omega, \tag{8}$$

$$u = u^{\partial\Omega}, \quad \text{on } \partial\Omega. \tag{9}$$

In this case, of course, we can find the axial Green's function associated with the convection operator in (8) and thus AGM can be applied on it. If we split domain Ω into Ω_1 and Ω_2 as in the left panel in Fig. 2, then the axial lines can be distributed as the right panel in Fig. 2 near the interface between two domains. Since we have the best approximations (5) and (6) of the solution, these equations enable us to merge the AGM solutions on both domains, Ω_1 and Ω_2. It is in fact an obvious advantage that there is no need for the conformity to axial lines across the interface. Assume $\mathbf{U} = (2, 3)$ and the exact solution $u(x, y)$ satisfying (8) and (9) in

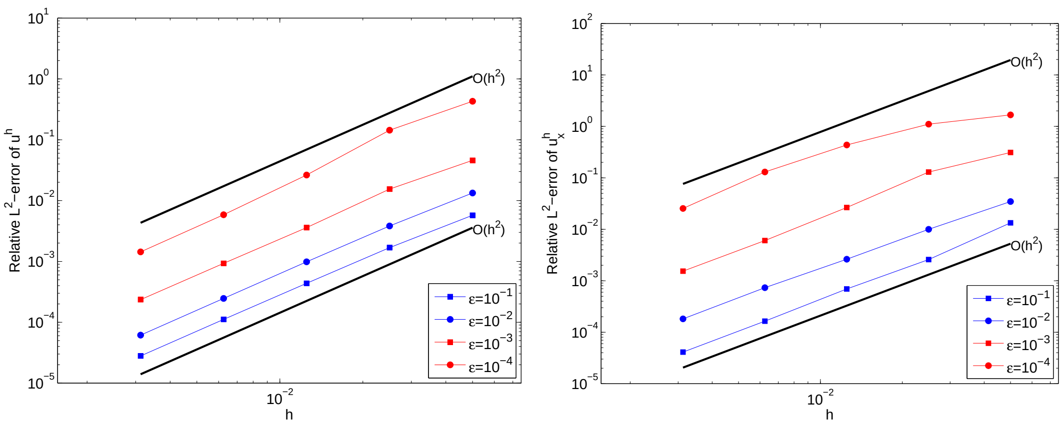

Figure 3: (left) $O(h^2)$-convergence for u^h in L^2-sense and (right) $O(h^2)$-convergence for the derivative of u^h in L^2-sense

$\Omega = [0,1] \times [0,1]$ as follows:

$$u(x,y) = 16x(10x)y(1-y)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(2/\sqrt{\epsilon}(0.25^2 - (x-0.5)^2 - (y-0.5)^2)\right)$$

which has interior steep layer. On the axial lines in Fig. 2, we obtain the second order convergence for the numerical solution u^h and its $x-$derivative u^h_x which are illustrated in Fig. 3.

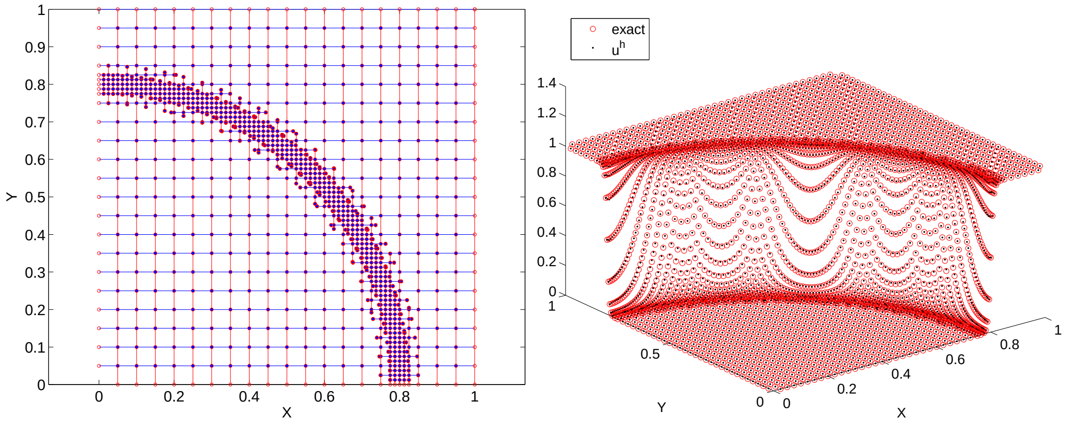

Figure 4: (left) Refined axial lines and (right) the corresponding numerical solution

In Fig. (4), two types of axial lines are drawn in the left panel, one is refinement near the steep interior layer of the solution and the other is coarse in a slowly varying region. Both the exact solution u and the computed solution u^h are depicted in the right panel of Fig. 4.

Conclusions

We present the axial Green' function method called the AGM which has two marked features. Firstly, arbitrarily distributed axial lines are available for the numerical computation without any degradation of accuracy. Second, the axial Green's function can be found easily compared to the multi-dimensional one. Using these features, we can devise an adaptive refinement of axial lines for the purpose of the effective computations. We expect that it can be applied to 3-dimensional problems as well.

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