Semilocal convergence of a parameter based iterative method for operator with bounded second derivative

†P.Maroju[1], R.Behl[2] and S.S.Motsa[3]

[1,2,3]Department of Mathematics, Statistics and Computer science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa

†Corresponding author: maroju.prashanth@gmail.com

Abstract

A parameter based set of third order iterative method and the semilocal convergence analysis of this methods using majorizing sequence approach for solving nonlinear equations in Banach spaces is investigated by Ezquerro and Hernandez [4]. This method is a weighted mean between the Chebyshev and the Halley methods, the weight being α and $1-\alpha$, where $\alpha \in \mathbb{R}$. A convergence theorem and corresponding error bounds provided. We have recurrence relation approach to discuss the semilocal convergence of iterative methods. This is motivated us to discuss the semilocal convergence. In this paper, mainly we focus on to discuss the semilocal convergence of parameter based iterative method developed by [4] using recurrence relations approach under the assumption that F'' is bounded and a punctual condition. Also, we established the R-order of convergence and provided some a priori error bounds. Finally, we discuss some numerical examples that where the Smale-like theorem fails but our bounded condition satisfy. We calculate the existence and uniqueness region for the Numerical examples. Also, we calculate the error bounds for parameter $\alpha = 0, 1, 2$. We observed that the existence region obtained by our approach is superior than Ezquerro and Hernandez [4] for each value of parameter $\alpha = 0, 1, 2$.

Keywords: The Halley's method, The Convex acceleration of Newton's method, A Continuation method, Banach space, Lipschitz condition, Fréchet derivative.

Introduction

Let $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain Ω and X, Y Banach spaces. In many years passed, one of the main problem in numerical analysis is to solve the nonlinear equation

$$F(x) = 0. \tag{1}$$

Many scientific and engineering problems, Kinetic theory of gases, elasticity, applied mathematics can be brought in the form of a nonlinear equation (1) and solved by using iterative methods. Newton in 1669 and Raphson in 1690 was proposed a procedure for solving nonlinear equation (1). Now, this method is called Newton's method or Newton-Raphson method and it is a central technique for solving nonlinear equations. The Newton's method is quadratically convergent. Basic results concerning that the semilocal convergence of Newton's method, the error estimates and the existence and uniqueness of solution are given by Kantorovich theorem. Kantrovich [9] established two different approaches to provide the proof of his theorem. Those are majorizing sequences and recurrence relations approaches.

Methods using higher order derivatives may be advantageous for special types of problems, if it is not particularly expensive to evaluate the involved derivatives in these methods. The well-known third-order methods of this type are Chebyshev, the Halley and the Super-Halley methods. These methods are of third order and can be successfully applied to solve (1). We have three different ways to study the convergence analysis of iterative methods. In the first technique, the convergence analysis have been studied under the assumption that first/second order Fréchet derivative satisfies Lipschitz/Hölder/ω-continuity conditions. This type of convergence analysis discussed by [2][3][7] using recurrence relations approach. This technique developed by these authors is an extension of technique followed by kantorovich and other authors [9][12] to study the Newton's method. In second technique, Smale [13] obtained the convergence of Newton's method for analytic maps from data at one point instead of Lipschitz continuity condition. Another technique is to discuss the convergence of (1) assume that F'' is bounded and a punctual condition, instead of Lipschitz continuity condition. Gutierrez and Hernandez [8] discussed the convergence analysis of third order iterative method under the assumption that F'' is bounded and a punctual condition.

Continuation, embedding or homotopy methods have long served as useful theoretical tools in modern mathematics. According to the basic idea of continuation methods [10][1], a homotopy $\alpha G(x) + (1-\alpha)H(x)$, where $\alpha \in [0,1]$, can be defined between two operators $G(x)$ and $H(x)$. Prashnath and Gupta [11] studied the semilocal convergence of continuation method between the Chebyshev and the Super-Halley methods by using recurrence relations approach. J.A.Ezquerro et.al [4][5][6] discussed the convergence analysis of continuation method between different third order iterative methods namely the Chebyshev, the Halley and the Super-Halley methods using majorizing sequence approach. Based on this idea, uniparametric family of iteration between the Chebyshev and the Halley's method derived by Ezquerro and Hernandez [4] is

$$\left.\begin{array}{l} x_{\alpha,n+1} = x_{\alpha,n} - [I + \frac{1}{2}L_F(x_{\alpha,n})G_\alpha(x_{\alpha,n})]F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}) \\ G_\alpha(x_{\alpha,n}) = I + \frac{\alpha}{2}L_F(x_{\alpha,n})J(x_{\alpha,n}) \\ J(x_{\alpha,n}) = (I - \frac{1}{2}L_F(x_{\alpha,n}))^{-1} \\ L_F(x_{\alpha,n}) = F'(x_{\alpha,n})^{-1}F''(x_{\alpha,n})F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}). \end{array}\right\} \quad (2)$$

This method (2) is parameter based method of order three which contain both methods for specific choice of the parameter. For $\alpha = 0$ the family mentioned above reduces to the Chebyshev method and for $\alpha = 1$ we get the Halley method. Ezquerro and Hernandez [4] discussed the convergence of this method using majorizing sequence approach under the assumptions that the second order Fréchet derivative satisfies Lipschitz continuity condition. Until now, we know that convergence of these methods is established assuming that the second order derivative F'' satisfies a Lipschitz continuity condition.

The main goal of this paper is to discuss the semilocal convergence of (2) using recurrence relation approach. We assume that F'' is bounded and a punctual condition instead of Lipschitz continuity condition. An existence-uniqueness theorem is given. We have also derived a closed form of error bounds in terms of parameter $\alpha \in \mathbb{R}$. We given some numerical applications to demonstrate our approach.

We end this section briefly by describing the organization of this paper. Section 1, is the introduction. In Section 2, the recurrence relations are derived. The a convergence theorem with the existence and uniqueness ball and error estimates for the solution is established in Section 3. In Section 4, two numerical examples are worked out to demonstrate the efficacy of our approach and the results obtained are compared with the results obtained in [4]. Finally, conclusions from the section 5.

Recurrence relations for the method

Let us suppose that $\Gamma_{\alpha,0} = F'(x_{\alpha,0})^{-1} \in \mathcal{L}(X, y)$ exists at some $x_{\alpha,0} \in \Omega$, where $\mathcal{L}(X, Y)$ is the set of bounded linear operators from Y into X. Moreover, we assume that following assumptions:

$$\left.\begin{array}{l} (i)\ \|\Gamma_{\alpha,0}\| = \|F'(x_{\alpha,0})^{-1}\| \leq \beta, \\ (ii)\ \|F'(x_{\alpha,0})^{-1}F(x_{\alpha,0})\| \leq \eta, \\ (iii)\ \|F''(x)\| \leq M,\ \forall\, x \in \Omega, \end{array}\right\} \tag{3}$$

Let us denote $a = M\beta\eta$. Then for $\alpha \in \mathbb{R}$ define the following real sequences for $n = 0, 1, 2, \ldots$

$$a_0 = 1, b_0 = 1, c_0 = a, d_0 = \frac{(\alpha-1)a^2+4}{2(2-a)}$$

$$a_{n+1} = \frac{a_n}{1 - a a_n d_n}, \quad b_{n+1} = \frac{a a_{n+1} d_n^2}{2}\left[1 + \frac{4 + c_n(2\alpha - 4) - (\alpha-1)c_n^2}{(2 + c_n + (\alpha-1)c_n^2)^2}\right]$$

$$c_{n+1} = a a_{n+1} b_{n+1}, \quad d_{n+1} = \left(\frac{2 + c_{n+1} + (\alpha-1)c_{n+1}^2}{2 - c_{n+1}}\right) b_{n+1}.$$

Let $\{x_{\alpha,n}\}$ a sequence of family. Based on these sequences, we now prove the following inequalities

(I) $\|\Gamma_{\alpha,n}\| = \|F'(x_{\alpha,n})^{-1}\| \leq a_n\beta$.

(II) $\|\Gamma_{\alpha,n}F(x_{\alpha,n})\| \leq b_n\eta$.

(III) $\|L_F(x_{\alpha,n})\| \leq c_n$.

(IV) $\|x_{\alpha,n+1} - x_{\alpha,n}\| \leq d_n\eta$.

The conditions (I), (II) and (III) for $n = 0$ hold from the assumptions (i), (ii) and

$$\|L_F(x_{\alpha,0})\| \;=\; \|F'(x_{\alpha,0})^{-1}F(x_{\alpha,0})F'(x_{\alpha,0})^{-1}F''(x_{\alpha,0}\| \leq M\beta\eta = a = c_0 < 1.$$

Using Banach Lemma, this gives

$$\|(I - \frac{1}{2}L_F(x_{\alpha,0}))^{-1}\| \leq \frac{1}{1 - \frac{1}{2}\|L_F(x_{\alpha,0})\|} = \frac{1}{1 - \frac{c_0}{2}} = \frac{1}{1 - \frac{a}{2}} = \frac{2}{2-a}.$$

From

$$G_\alpha(x_{\alpha,0}) = I + \frac{\alpha}{2}L_F(x_{\alpha,0})J(x_{\alpha,0})$$

we get

$$\|G_\alpha(x_{\alpha,0})\| \;\leq\; 1 + \frac{\alpha}{2}\|L_F(x_{\alpha,0})\|\|J(x_{\alpha,0})\| \leq \frac{2 + (\alpha-1)a}{(2-a)}.$$

Using (2) and condition (II) we get

$$\|x_{\alpha,1} - x_{\alpha,0}\| \leq \left[\frac{4 + (\alpha-1)a^2}{2(2-a)}\right]\eta \leq d_0\eta.$$

Hence, the condition (IV) also hold true for $n = 0$. Let us assume that the conditions (I)-(IV) hold true for $n = k$. To prove that they also hold true for $n = k+1$, we use $x_{\alpha,k} \in \Omega,\ c_k < 1$ and $aa_kd_k < 1$ to get $\|I - \Gamma_{\alpha,k}F'(x_{\alpha,k})\| \leq aa_kd_k < 1$. Now, by using Banach's theorem, we find that $\Gamma_{\alpha,k+1} = F'(x_{\alpha,k+1})^{-1}$ exists and

$$\begin{aligned} \|\Gamma_{\alpha,k+1}\| &\leq \frac{\|\Gamma_{\alpha,k}\|}{1 - \|I - \Gamma_{\alpha,k}F'(x_{\alpha,k}\|} \\ &\leq \frac{a_k\beta}{1 - aa_kd_k} = a_{k+1}\beta. \end{aligned} \tag{4}$$

Now from (2),

$$\begin{aligned} F(x_{\alpha,k+1}) &= \int_0^1 [F'(x_{\alpha,k} + t(x_{\alpha,k+1} - x_{\alpha,k})) - F'(x_{\alpha,k})](x_{\alpha,k+1} - x_{\alpha,k})dt \\ &\quad -\frac{1}{2}F''(x_{\alpha,k})F'(x_{\alpha,k})^{-1}F(x_{\alpha,k})G_\alpha(x_{\alpha,k})F'(x_{\alpha,k})^{-1}F(x_{\alpha,k}) \end{aligned}$$

From this,

$$\|F(x_{\alpha,k+1})\| \leq \frac{M\eta^2d_k^2}{2} + \frac{M\eta^2b_k^2(2 + (\alpha - 1)c_k)}{2(2 - c_k)} \tag{5}$$

and

$$\begin{aligned} \|\Gamma_{\alpha,k+1}F(x_{\alpha,k+1})\| &\leq \|\Gamma_{\alpha,k+1}\|\|F(x_{\alpha,k+1})\| \\ &\leq a_{k+1}\beta M\eta^2\Big[\frac{d_k^2}{2} + \frac{b_k^2(2 + (\alpha - 1)c_k)}{2(2 - c_k)}\Big] \\ &= \frac{aa_{k+1}d_k^2}{2}\Big[1 + \frac{b_k^2(2 + (\alpha - 1)c_k)}{d_k^2(2 - c_k)}\Big]\eta \\ &= \frac{aa_{k+1}d_k^2}{2}\Big[1 + \frac{4 + c_k(2\alpha - 4) - (\alpha - 1)c_k^2}{(2 + c_k + (\alpha - 1)c_k^2)^2}\Big]\eta \end{aligned}$$

This gives

$$\|\Gamma_{\alpha,k+1}F(x_{\alpha,k+1})\| \leq b_{k+1}\eta. \tag{6}$$

Also from,

$$\begin{aligned} \|L_F(x_{\alpha,k+1})\| &\leq \|F'(x_{\alpha,k+1})^{-1}\|\|F'(x_{\alpha,k+1})^{-1}F(x_{\alpha,k+1})\|\|F''(x_{\alpha,k+1})\| \\ &\leq a_{k+1}\beta b_{k+1}\eta M = M\beta\eta a_{k+1}b_{k+1} = aa_{k+1}b_{k+1} \end{aligned}$$

we get

$$\|L_F(x_{\alpha,k+1})\| \leq c_{k+1} \tag{7}$$

Again using,

$$\begin{aligned}\|x_{\alpha,k+2}-x_{\alpha,k+1}\| &\leq [1+\frac{1}{2}\|L_F(x_{\alpha,k+1})\|\|G_\alpha(x_{\alpha,k+1})]\|\Gamma_{\alpha,k+1}F(x_{\alpha,k+1})\| \\ &= \Big[\frac{2+c_{k+1}+(\alpha-1)c_{k+1}^2}{(2-c_{k+1})}\Big]b_{k+1}\eta\end{aligned}$$

we get

$$\|x_{\alpha,k+2}-x_{\alpha,k+1}\| \leq d_{k+1}\eta. \tag{8}$$

From (4),(6), (7) and (8) conclude that the conditions (I)-(IV) hold true for $n=k+1$.

Convergence Analysis

In this section, discuss the properties of real sequences and establish a convergence theorem and the existence and uniqueness region along with an estimation of the error bounds for the method (2). First at all we give a technical lemma including the results concerning one and two variable functions that we are going to need. We omit the proof to the reader could get it patiently but without any difficulty.

Lemma 1 *The following recurrence relation holds for the sequence* $\{c_n\}$.

$$c_{n+1}=\frac{c_n^2}{2}\Big[\frac{c_n^4(\alpha^2-2\alpha+1)+c_n^3(2\alpha-2)+c_n^2(3\alpha-2)+2c_n\alpha+8}{(2-3c_n-c_n^2-(\alpha-1)c_n^3)^2}\Big]$$

Lemma 2 *Let* $a_0=0.291481$ *be the smallest positive root of polynomial* $-2x^6+5x^5+8x^4-22x^3-10x^2+32x-8=0$ *and define the functions*

$$h(x)=\frac{-2-11a+10a^2+6a^3-4a^4+\sqrt{4+76a-111a^2+52a^3-8a^4}}{2(a^3-a^4)},$$

$$H(x,y)=\frac{y^4(x^2-2x+1)+x^3(2x-2)+x^2(3x-2)+2xy+8}{(2-3y-y^2-(x-1)y^3)^2},$$

$$g_\alpha(x)=\frac{(2-x)}{2-3x-x^2-(\alpha-1)x^3},$$

$$f_\alpha(x)=\frac{2+x+(\alpha-1)x^2}{(2-x)}.$$

then

- (i) $h(x)$ *is a decreasing function.*
- (ii) $H(x,y)$ *is increasing as a functions of* y *in* $(0,a_0]$ *and* $0\leq x\leq h(y)$.
- (iii) $f_\alpha(x)$ *and* $g_\alpha(x)$ *are increasing for all* $\alpha\geq 0$.

Proof: This proof is simple then omitted for the readers.

Lemma 3 Let $0 < a \leq a_0$ and $0 \leq \alpha \leq h(a)$, then the sequence $\{c_n\}$ is decreasing.

Proof. This Lemma can be proved by induction. From Lemma 2, $c_{n+1} \leq c_n$ if

$$\frac{c_n}{2}\Big[\frac{c_n^4(\alpha^2-2\alpha+1)+c_n^3(2\alpha-2)+c_n^2(3\alpha-2)+2c_n\alpha+8}{(2-3c_n-c_n^2-(\alpha-1)c_n^3)^2}\Big] \leq 1, n \geq 0$$

for $n = 0$, we get

$$a^5(\alpha^2-2\alpha+1)+a^4(2\alpha-2)+a^3(3\alpha-2)+2a^2\alpha+8a \leq 2(2-3a-a^2-(\alpha-1)a^3)^2$$

This gives,

$$(-2a^6+a^5)\alpha^2+(4a^6-6a^5-10a^4+11a^3+2a^2)\alpha-2a^6+5a^4+8a^4-22a^3-10a^2+32a-8 \geq 0$$

This hold true for, $0 \leq \alpha \leq h(a)$. Hence $c_1 \leq c_0$. Let us assume that $c_k \leq c_{k-1} \ldots \leq c_1 \leq c_0$. Since, $h(x)$ is a decreasing function, so that $\alpha \leq h(a) = h(c_0) \leq h(c_k)$. Hence, $c_{k+1} \leq c_k$.

Lemma 4 *Under the hypothesis of Lemma ,$aa_nd_n < 1$ for $n \geq 0$ and $\{a_n\}$ is an increasing sequence.*

Proof. We have,

$$aa_nd_n = \frac{c_n(2+c_n+(\alpha-1)c_n^2)}{(2-c_n)}.$$

Then, $aa_nd_n < 1$ if $\alpha < q(c_n)$, where $q(x) = (x^3-x^2-3x+2)/x^3$. As $q(x)$ is decreasing and $c_n \leq c_0, q(c_n) \geq q(c_0)$. Besides, $\alpha < h(a)$ for $a \in (0, a_0]$. Indeed $q(a) - h(a) > 0$. Hence, $aa_nd_n < 1$ for $n \geq 0$. Finally, $a_0 = 1$, $a_1 = \frac{a_0}{1-aa_0d_0} > a_0 = 1$ and inductively, $a_{n+1} = a_n/(1-aa_nd_n) \geq a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0$.

Lemma 5 *Under the assumptions, $0 < a \leq a_0$ and $0 \leq \alpha \leq h(a)$. Then $c_{n+1} \leq \gamma^{2^n}\frac{c_0}{\gamma}$, where $\gamma = c_1/c_0$. Also the sequence $\{c_n\}$ converges to 0 and $\sum_{n=0}^{\infty} c_n < \infty$.*

Proof. First we prove the first part of Lemma. Let $c_1 = \gamma c_0$, with $\gamma < 1$. We prove that $c_n \leq \gamma c_{n-1}$ implies $c_{n+1} \leq \gamma^2 c_n$. From Lemma 1 we get

$$c_{n+1} = \frac{c_n^2}{2}H(\alpha, c_n) \leq \frac{\gamma^2 c_{n-1}^2}{2}H(\alpha, c_n).$$

As $H(\alpha, y)$ is increasing in the second variable and $c_n < c_{n-1}$, we get

$$c_{n+1} = \frac{c_n^2}{2}H(\alpha, c_n) \leq \gamma^2 c_n.$$

Then we have $c_{n+1} \leq \gamma^{2^n} c_n$ and using this inequality, $c_n \leq \gamma^{2^n} c_0/\gamma$. As $\gamma < 1$, the first part proved. The second part of the proof is simple and omitted for readers. Hence the Lemma is proved.

Lemma 6 *The sequence $\{a_n\}$ is bounded above, that is, there exists a constant $M > 0$ such that $a_n \leq M \quad \forall n \in \mathbb{N}$*

Proof. From $a_{n+1} = \frac{a_n}{1-aa_nd_n}$ and $g_\alpha(c_n) = \frac{(2-c_n)}{2-3c_n-c_n^2-(\alpha-1)c_n^3}$ which gives,

$$a_{n+1} = a_n\Big[1 + c_ng_\alpha(c_n)\Big] = \Pi_{k=0}^n\Big[1 + c_kg_\alpha(c_k)\Big]$$

Taking log on both sides, we get

$$\log a_{n+1} = \Sigma_{k=0}^n \log(1 + c_kg_\alpha(c_k)) \le \Sigma_{k=0}^n c_kg_\alpha(c_k) < \infty.$$

Hence, $\{a_n\}$ is a bounded sequence.

Lemma 7 *The sequence $\{d_n\}$ is a cauchy sequence and satisfies the condition $d_n \le \gamma^{2^n-1}d_0$ for $0 < a \le a_0$.*

Proof. From

$$d_n = f_\alpha(c_n)\frac{c_n}{aa_n}, \quad \text{where, } f_\alpha(c_n) = \frac{2+c_n+(\alpha-1)c_n^2}{(2-c_n)}.$$

Since $a_n > 1$, so we get, $d_n \le c_nf_\alpha(c_n)/a \le \gamma^{2^n-1}d_0$ for $\gamma < 1$. Thus,the sequence $\{d_n\}$ converges to 0. Hence it is a cauchy sequence.

Theorem 1 *Let X and Y be two Banach spaces and let $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable on a non-empty open convex subset Ω. Assume that $\Gamma_{\alpha,0} = F'(x_{\alpha,0})^{-1}$ exist at some $x_{\alpha,0} \in \Omega$ and the assumptions (i)-(iii) are satisfied. Let us denote $a_0 = M\beta\eta$. Suppose that $0 < a \le a_0 = 0.291481$ and $0 \le \alpha \le h(a)$, where $h(x)$ is the function defined in Lemma 1. Then, if $\overline{\mathcal{B}}(x_{\alpha,0}, r\eta) = \{x \in X : \|x - x_{\alpha,0}\| \subseteq \Omega$, where, $r = \Sigma_{n=0}^\infty d_n$, the sequence $\{x_{\alpha,n}\}$ defined in (2) and starting at $x_{\alpha,0}$ converge to a solution x^* of the equation (1). In this case the solution x^* and the iterates $x_{\alpha,n}$ lies in $\overline{\mathcal{B}}(x_{\alpha,0}, r\eta)$, and the solution x^* is unique in the open ball $B(x_{\alpha,0}, 2/M\beta - r\eta)$. Further, the error estimate of the method in terms of real sequence $\{d_n\}$ is given by*

$$\|x^* - x_{\alpha,n+1}\| \le \sum_{k=n+1}^{\infty} d_k\eta.$$

Proof. For $0 < a < a_0$, $0 \le \alpha < h(a)$ and using above Lemmas, the sequence $\{x_{\alpha,n}\}$ converge to the solution. For $\alpha = h(a)$, we have $c_n = c_0 = a$, for $n \ge 0$, From

$$a_{n+1} = \frac{a_n}{1 - aa_nd_n}$$

and

$$d_n = \Big[\frac{2 + c_n + (\alpha - 1)c_n^2}{2 - c_n}\Big]\frac{c_n}{aa_n}$$

we get

$$a_{n+1} = a_n\Big[1 + \frac{(2 - c_0)}{2 - 3c_0 - c_0^2 - (\alpha - 1)c_0^3}\Big]$$

Taking, $w = \left[1 + \frac{(2-c_0)}{2-3c_0-c_0^2-(\alpha-1)c_0^3}\right]$. This can be written as $a_{n+1} = wa_n = w^{n+1}a_0$. Since $a_0 = 1$, this gives $a_{n+1} = w^{n+1}$ and

$$\begin{aligned} d_n &= \left[\frac{2+c_n+(\alpha-1)c_n^2}{2-c_n}\right]\frac{c_n}{aa_n} \\ &= \left[\frac{2+c_0+(\alpha-1)c_0^2}{2-c_0}\right]\frac{c_0}{aa_0} \\ &= \frac{1}{w^n}\left[\frac{2+c_0+(\alpha-1)c_0^2}{2-c_0}\right]\frac{c_0}{aa_0} \end{aligned}$$

Hence, $\lim_{n\to\infty} d_n = 0$. Thus, $\{d_n\}$ is a cauchy sequence. From condition (IV), we get $\{x_{\alpha,n}\}$ is also a cauchy sequence and hence there exists a x^* such that $\lim_{n\to\infty} x_{\alpha,n} = x^*$. Now from the equation (5) , we get

$$\|F(x_{\alpha,n+1})\| \leq \frac{M\eta^2}{2}\left[d_n^2 + \frac{b_n^2(2+(\alpha-1)c_n)}{2-c_n}\right], \tag{9}$$

the limit of the sequence $\{b_n\}$ and $\{d_n\}$ is 0 and the continuity of F, we prove that $F(x^*) = 0$. Thus, x^* is a solution of equation (1). Also

$$\begin{aligned} \|x_{\alpha,n+1} - x_0\| &\leq \|x_{\alpha,n+1} - x_{\alpha,n}\| + \|x_{\alpha,n} - x_{\alpha,n-1}\| + ... + \|x_{\alpha,1} - x_{\alpha,0}\| \\ &\leq \sum_{k=0}^{n} d_k\eta \\ &\leq r\eta \end{aligned}$$

This gives $x_{\alpha,n} \in \overline{\mathcal{B}}(x_{\alpha,0}, r\eta)$. Now taking limit as $n \to \infty$, we get $\|x^* - x_{\alpha,0}\| \leq r\eta$ and hence $x^* \in \overline{\mathcal{B}}(x_{\alpha,0}, r\eta)$. Also for every $m \geq n+1$, we get

$$\begin{aligned} \|x_{\alpha,m} - x_{\alpha,n+1}\| &\leq \|x_{\alpha,m} - x_{\alpha,m-1}\| + \|x_{\alpha,m-1} - x_{\alpha,m-2}\| + ... + \|x_{\alpha,n+2} - x_{\alpha,n+1}\| \\ &\leq \sum_{k=n+1}^{\infty} d_k\eta < r\eta \end{aligned}$$

by taking $m \to \infty$, we get $\|x^* - x_{\alpha,n+1}\| \leq \sum_{k=n+1}^{\infty} d_k\eta < r\eta$.

To prove the uniqueness of the solution, if y^* be the another solution of (1) then we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*)$$

Clearly, $y^* = x^*$, if $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. This follows from

$$\begin{aligned} \|\Gamma_{\alpha,0}\|\|\int_0^1[F'(x^* + t(y^* - x^*)) - F'(x_{\alpha,0}]dt\| &\leq M\beta\int_0^1 \|x^* + t(y^* - x^*) - x_{\alpha,0}\|dt \\ &\leq M\beta\int_0^1 (1-t)\|x^* - x_{\alpha,0}\| + t\|y^* - x_{\alpha,0}\|dt \\ &\leq \frac{M\beta}{2}(r\eta + \frac{2}{k_1\beta} - r\eta) = 1 \end{aligned}$$

and by Banach's theorem. Thus, $y^* = x^*$.

Numerical Examples

Example 1 *Consider the function* $F(x) = 0$, *where,*

$$F(x) = 9x^{7/3} + 4x^2 - 36x + 9, \tag{10}$$

defined in $X = [-1, 1]$ *and initial approximation* $x_0 = 0$

Solution: From this, we observed that $F^{(k)}(x)$ does not defined at x_0 for $k \geq 3$. So Smale-like condition do not work. Hence, Using the assumptions (i)-(iii) for the initial value $x_0 = 0$, we get $\beta = 1/36$, $\eta = 1/4$, and $M = 36$. Hence, $a = M\beta\eta = 0.25 < a_0$ and we can take the real sequences defined in (2) for $0 \leq \alpha \leq h(a) = 2.12382$. We calculate the real sequences for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$ displayed in following Tables.

Table-1 : Real sequences for $\alpha = 0$

n	a_n	b_n	c_n	d_n	$\sum d_n$
0	1.00000	1.00000	0.25000	1.12500	1.12500
1	1.39130	0.360978	0.125558	0.406302	1.53130
2	1.62029	0.0598264	0.024234	0.0612762	1.59258
3	1.66153	0.0015232	0.00063271	0.00152416	1.59410
4	1.66258	9.64964e-007	4.01083e-007	9.64964e-007	1.59410
5	1.66258	3.87031e-013	1.60868e-013	3.87031e-013	1.59410
6	1.66258	6.22607e-026	2.58784e-026	6.22607e-026	1.59410
7	1.66258	1.6112e-051	6.6969e-052	1.6112e-051	1.59410
8	1.66258	1.07901e-102	4.48484e-103	1.07901e-102	1.59410
9	1.66258	4.83917e-205	2.01138e-205	4.83917e-205	1.59410
10	1.66258	0.	0.	0.	1.59410

Table-2 : Real sequences for $\alpha = 1$

n	a_n	b_n	c_n	d_n	$\sum d_n$
0	1.	1.	0.25000	1.14286	1.14286
1	1.40000	0.386596	0.135309	0.442702	1.58556
2	1.6567	0.0737824	0.0305588	0.0760721	1.66163
3	1.71059	0.00241948	0.00103469	0.00242198	1.66406
4	1.71237	2.50924e-006	1.07419e-006	2.50924e-006	1.66406
5	1.71237	2.6954e-012	1.15388e-012	2.6954e-012	1.66406
6	1.71237	3.11017e-024	1.33144e-024	3.11017e-024	1.66406
7	1.71237	4.141e-048	1.77273e-048	4.141e-048	1.66406
8	1.71059	7.34087e-096	3.14257e-096	7.34087e-096	1.66406
9	1.71237	2.30692e-191	9.87574e-192	2.30692e-191	1.66406
10	1.71237	0.	0.	0.	1.66406

Table-3 : Real sequences for $\alpha = 2$

n	a_n	b_n	c_n	d_n	$\sum d_n$
0	1.	1.	0.25	1.16071	1.16071
1	1.40881	0.411943	0.145087	0.48106	1.64177
2	1.69619	0.0906748	0.0384504	0.0942979	1.73607
3	1.76684	0.00385091	0.00170098	0.00385747	1.73993
4	1.76986	6.57829e-006	2.91066e-006	6.5783e-006	1.73993
5	1.76986	1.91473e-011	8.472e-012	1.91473e-011	1.73993
6	1.76986	1.62216e-022	7.17749e-023	1.62216e-022	1.73993
7	1.76986	1.1643e-044	5.15163e-045	1.1643e-044	1.73993
8	1.76986	5.99805e-089	2.65393e-089	5.99805e-089	1.73993
9	1.76986	1.59184e-177	7.04334e-178	1.59184e-177	1.73993
10	1.76986	0.	0.	0.	1.73993

From Table-1 for $\alpha = 0$ we get $r = \sum d_n = 1.59410$. So the existence and uniqueness solution of (10) are $\overline{\mathcal{B}}(x_{0,0}, 0.398525) \subseteq \Omega$, $\mathcal{B}(x_{0,0}, 1.60148) \bigcap \Omega$. From Table-2 for $\alpha = 1$ we get $r = \sum d_n = 1.66406$. So the existence and uniqueness solution of (10)respectively are $\overline{\mathcal{B}}(x_{1,0}, 0.416015) \subseteq \Omega$, $\mathcal{B}(x_{1,0}, 1.58399) \bigcap \Omega$. From Table-3 for $\alpha = 2$ we get $r = \sum d_n = 1.73993$. So the solution of (10) exists in $\overline{\mathcal{B}}(x_{2,0}, 0.434983) \subseteq \Omega$ and unique in $\mathcal{B}(x_{2,0}, 1.56502) \bigcap \Omega$. However, solving (10) by using majorizing sequence [4], for $\alpha \in (-15, 2)$ we find that the solution exists in the ball $\overline{\mathcal{B}}(x_{\alpha,0}, 0.292893) \subseteq \Omega$ and unique in $\mathcal{B}(x_{\alpha,0}, 1.70711) \bigcap \Omega$. From this result, we can easily conclude that our existence region of solution is greater than the existence region obtained by majorizing sequences. Also, we calculated error bounds by our approach and with majorizing sequence approach [4] given in Table-4.

Table-4:Error bounds for $\alpha = 0$ and $\alpha = 1$

n	$\alpha = 0$	$\alpha = 1$	$\alpha = 0$ by [4]	$\alpha = 1$ by [4]
0	0.11727600	0.13030000	0.292893	0.292893
1	0.01570000	0.01962400	0.0144311	0.00717893
2	0.00038100	0.00060600	2.91523e-6	1.82202e-007
3	2.41241e-007	6.27312e-007	2.47752e-017	3.02432e-021
4	9.67577e-014	6.7385e-013	1.52073e-050	1.3831e-062
5	1.55652e-026	7.77542e-025	3.5169e-150	1.32291e-186
6	4.02801e-052	1.03525e-048	4.3498e-449	1.157605e-558

Example 2 *Let $X = C[0, 1]$ be the space of all continuous functions on the interval $[0, 1]$ and consider the H-equation called integral equation of Chandrasekhar*

$$F(x)(s) = 1 - x(s) + \frac{1}{4}x(s) \int_0^1 \frac{s}{s+t} x(t)dt \tag{11}$$

If we choose $x_0 = x_0(s) = s$ and the norm $\|x\| = \max_{s \in [0,1]} |x(s)|$. Then we get, $M = 0.3465, \beta = 1.5304$ and $\eta = 0.2652$. Hence, we get $a = M\beta\eta = 0.1406312 < a_0$. Also, we can take the real sequence (2) for $0 \leq \alpha \leq h(a) = 46.1089$. The real sequences for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$ is given in following Table-5, Table-6 and Table-7. For $\alpha = 0$, from the Table-5 the solution of (11) exists in the ball $\overline{\mathcal{B}}(x_{0,0}, 0.330869)$ and is unique in the ball $\mathcal{B}(x_{0,0}, 3.4407)$. For $\alpha = 1$, from the Table-6 the solution of (11) exists in the ball $\overline{\mathcal{B}}(x_{1,0}, 0.33407)$ and is unique in the ball $\mathcal{B}(x_{1,0}, 3.4375)$. For $\alpha = 2$, from the Table-7 the solution of (11) exists in the ball $\overline{\mathcal{B}}(x_{2,0}, 0.337268)$ and is unique in the ball $\mathcal{B}(x_{2,0}, 3.4343)$. However, solving (11) by using majorizing sequence [4], for $\alpha \in (-15, 2$ we find that the solution exists in the ball $\overline{\mathcal{B}}(x_{\alpha,0}, 0.287047) \subseteq \Omega$ and is unique in $\mathcal{B}(x_0, 3.48452)$. From this result, we can easily conclude that our existence region of solution is greater than the existence region obtained by majorizing sequences.

Table-5 : Real sequences for $\alpha = 0$

n	a_n	b_n	c_n	d_n	$\sum d_n$
0	1.00000	1.00000	0.140631	1.07032	1.07032
1	1.17719	0.167709	0.0277641	0.172365	1.24268
2	1.21177	0.00492798	0.000839789	0.00493212	1.24762
3	1.21279	4.14542e-006	7.07026e-007	4.14543e-006	1.24762
4	1.21279	2.93093e-012	4.99887e-013	2.93093e-012	1.24762
5	1.21279	1.46513e-024	2.49887e-025	1.46513e-024	1.24762
6	1.21279	3.66117e-049	6.24435e-050	3.66117e-049	1.24762
7	1.21279	2.28616e-098	3.89919e-099	2.28616e-098	1.24762
8	1.21279	8.91419e-197	1.52037e-197	8.91419e-197	1.24762
9	1.21279	0.	0.	0.	1.24762

Table-6 : Real sequences for $\alpha = 1$

n	a_n	b_n	c_n	d_n	$\sum d_n$
0	1.000000	1.000000	0.140631	1.07563	1.07563
1	1.17823	0.173644	0.028772	0.178713	1.25434
2	1.21418	0.00533859	0.000911574	0.00534346	1.25969
3	1.21529	4.87651e-006	8.33434e-007	4.87652e-006	1.25969
4	1.21529	4.06426e-012	6.94615e-013	4.06426e-012	1.25969
5	1.21529	2.8231e-024	4.8249e-025	2.8231e-024	1.25969
6	1.21529	1.36212e-048	2.32796e-049	1.36212e-048	1.25969
7	1.21529	3.17095e-097	5.41941e-098	3.17095e-097	1.25969
8	1.21529	1.71847e-194	2.937e-195	1.71847e-194	1.25969
9	1.21529	0.	0.	0.	1.25969

Table-7 : Real sequences for $\alpha = 2$

n	a_n	b_n	c_n	d_n	$\sum d_n$
0	1.	1.	0.140631	1.08095	1.08095
1	1.17927	0.179515	0.029771	0.185021	1.26597
2	1.2166	0.00576852	0.0009869	0.005774	1.27175
3	1.2178	5.70729e-006	9.77435e-007	5.7073e-006	1.27175
4	1.2178	5.57852e-012	9.55382e-013	5.57852e-012	1.27175
5	1.2178	5.32961e-024	9.12754e-025	5.32961e-024	1.27175
6	1.2178	4.86462e-048	8.3312e-049	4.86462e-048	1.27175
7	1.2178	4.05281e-096	6.94088e-097	4.05281e-096	1.27175
8	1.2178	2.81301e-192	4.81759e-193	2.81301e-192	1.27175
9	1.2178	0.	0.	0.	1.27175

Conclusions

In this paper, we discussed the semilocal convergence of parameter based iterative method under the assumption that second order Fréchet derivative satisfies bounded condition instead of Lipschitz continuity condition. The analysis discussed using recurrence relation approach. Based on this approach, the existence and uniqueness region with priori error bounds established. Finally, Numerical examples are worked out to demonstrate our approach. we observed that our approach have more superior error bounds than the other approach [4].

References

[1] Allgower , E.L.,Georg, K. (1990) Numerical Continuation Methods, an Introduction, *Springer-Verlag, New York.*

[2] Candela, V, Valencia, A. M. (1990) Recurrence Relation for Rational Cubic Methods I: The Halley Method, *Computing* **44**, 169-184.

[3] Candela, V, Valencia, A. M. (1990) Recurrence Relation for Rational Cubic Methods II: The Chebyshev Method, *Computing* **45**, 355-367.

[4] Ezquerro, J.A, Hernández, M. A. (1999) On a class of iteration containing the chebyshev and the Halley methods, *Publ.Math.Debrecen* **54**, 403-415.

[5] Ezquerro, J.A, Gutiérrez, J.M., Hernández, M. A. (1997) A Construction Procedure of Iterative Methods with Cubical Convergence, *Journal of Applied Mathematics and Computation* **85**, 181-199.

[6] Ezquerro, J.A, Hernández, M. A. (2003) A New class of third order methods in Banach spaces, *Journal of Applied Mathematics and Computation* **31**, 181-199.

[7] Gutiérrez, J.M., Hernández, M. A. (1998) Recurrence Relations for the Super-Halley Method, *Journal of Computer Math.Applications* **36**, 1-8.

[8] Hernández, M. A.,Gutiérrez, J.M. (1997) Third-order iterative methods for operators with bounded second derivative, *Journal of Computational and Applied Mathematics* **82**, 171-183.

[9] Kantorovich , L.V.,Akilov, G.P. (1982) Functional Analysis. *Pergamon Press, Oxford.*

[10] Ortega , J.M.,Rheinboldt.W.C (1970) Iterative solution of nonlinear equations in several variables, *Academic Press*.

[11] Prashnath, M, Gupta, D. K. (2013) A Continuation method and its convergence for solving nonlinear equations in banach spaces, *International Journal of Computational Methods* **10**.

[12] Rall , L.B (1979) Computational Solution of Nonlinear Operator Equations, *Krieger, New York.*

[13] Smale,S. (1986) Newton's method estimates from data at one point, *Proc. Conf. in Honor of Gail Young*.