The numerical manifold method for two-dimensional transient heat conduction problems

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Abstract

Due to the use of two cover systems, i.e., the mathematical cover system and the physical cover system, the numerical manifold method (NMM) is able to solve both continuous and discontinuous problems within the same framework. In the present paper, the NMM is developed to analyze unsteady heat conduction problems in two-dimensional settings. The NMM discrete equations are derived using the weighted residual method in Galerkin form. The spatial integration is performed through triangulation and Gauss quadrature while time integration is realized by the backward Euler scheme. The proposed approach is verified through a typical numerical example.

Keywords: Numerical manifold method (NMM), Two-dimensional heat conduction, Transient, Temperature

Introduction

In the past two decades, considerable efforts have been put on to the development of the numerical manifold method (NMM) proposed by Shi [Shi (1991)]. The outstanding performance of the NMM originates from the use of finite cover concept. Benefiting from the use of dual cover systems, that is, the mathematical cover system and the physical cover system, the NMM is able to solve both continuous and discontinuous problems in a unified framework. The major highlights of the NMM can be summarized in the following aspects: (1) the mathematical cover system can be independent of both external and internal boundaries; (2) the local property of physical field can be manifested in essence or through the proper choice of cover functions; (3) Higher-order approximation can be achieved at a fixed mathematical cover system by the use of higher-order cover functions.

Since the advent, the NMM has been applied and developed to solve various problems in many fields. Tsay et al applied the NMM to predict crack growth trajectory combined with the local remeshing technique [Tsay et al. (1999)]. Chiou et al adopted the NMM to investigate mixed mode crack propagation together with the virtual crack extension method [Chiou et al. (2002)]. Li et al developed the enriched meshless manifold method to solve two-dimensional (2D) crack problems [Li et al. (2005)]. Terada et al applied the NMM (called finite cover method therein) to analysis progressive failure processes involving cohesive zone fracture in heterogeneous solids and structures [Terada et al. (2007)]. Kurumatani and Terada extended the NMM to crack simulations for quasi-brittle heterogeneous solids by using only a regular structured mathematical mesh [Kurumatani and Terada (2009)]. Ma and his co-authors tackled 2D complex crack problems using singular physical covers in the NMM [Ma et al. (2009)], and then they further studied multiple crack propagation problems [Zhang et al. (2010)]. Zhao et al applied the NMM to consider the microstructure influence of materials in plane micropolar elasticity [Zhao et al. (2010)]. An et al introduced weak-discontinuous physical covers to describe material discontinuities within the framework of NMM [An et al. (2011)]. Zhang and Zhang computed the SIFs on polygonal mathematical elements by the NMM [Zhang and Zhang (2012)]. Wu and Wong studied the effects of the friction and cohesion on the crack growth from a closed crack under compression with the NMM [Wu and Wong (2012)]. An et al solved 2D bimaterial interface crack problems by the NMM [An et al. (2013)]. Fan et al simulated the stress wave propagation through fracture rock with the NMM [Fan et al. (2013)]. Zhang and Ma investigated the fracture of functionally graded materials by the NMM [Zhang and Ma (2014)]. Zhang et al focused on 2D crack problems under thermo-mechanical loading [Zhang et al. (2014)]. Hu et al developed a discontinuous approach for the simulation of fluid flow in heterogeneous media by the NMM [Hu et al. (2015)]. Zheng et al proposed a mixed solution to the unconfined seepage problems with the NMM [Zheng et al. (2015)].Wang et al proposed a second-order NMM to study free surface flow containing inner drains [Wang et al. (2016)].

In the present paper, the NMM is further developed to study 2D unsteady heat conduction problems. To this end, the remaining of the paper is addressed as follows. Firstly, the governing equations and associated boundary and/or initial conditions for concerned problems are provided. Secondly, the NMM formulations for transient heat conduction analysis are derived; then, to verify the proposed method, a typical numerical example is tested. Finally, the corresponding conclusions are drawn.

Governing equations

Ignoring the heat source, the governing equations for transient heat conduction problems is [Prasad et al. (1996)]

$$\rho c \frac{\partial T(\mathbf{x},t)}{\partial t} + \nabla \mathbf{q}(\mathbf{x},t) = 0 \tag{1}$$

where ρ is the mass density and c is the specific heat at constant pressure. ∂ denotes partial derivative. $T(\mathbf{x},t)$ is the temperature with $\mathbf{x} \in \Omega$ (Ω denotes the physical domain) and t the time. The heat flux $\mathbf{q}$ is determined by the Fourier's law as $\mathbf{q} = -k\nabla T$ with k the thermal conductivity for isotropic material and ∇ the gradient operator.

The associated boundary conditions are

$$T(\mathbf{x},t) = \bar{T}(\mathbf{x},t) \qquad (\mathbf{x} \in \Gamma_T) \tag{2}$$

$$\mathbf{q}(\mathbf{x},t)\cdot\mathbf{n} = \bar{q}(\mathbf{x},t) \quad (\mathbf{x} \in \Gamma_q) \tag{3}$$

where Γ_T is the temperature boundary and Γ_q is the flux boundary. $\bar{T}$ and $\bar{q}$ are, respectively, the prescribed temperature and flux on corresponding boundary. $\mathbf{n}$ is the outward unit normal to the domain.

The initial condition for Eq. (1) is

$$T(\mathbf{x},0) = T_0 \quad (\mathbf{x} \in \Omega) \tag{4}$$

The NMM for unsteady heat conduction

A brief introduction of the NMM

In the NMM, to solve a given problem, the mathematical cover (MC) system is firstly built. Broadly speaking, the MC composed of mathematical elements can be of any shape and the MC system may be independent of all domain boundaries (including internal ones) but must be large enough to cover the whole domain. On each MC, a partition of unity (PU) [Melenk and Babuska (1996)] weight function is defined. Next, the physical cover (PC) system is formed by the intersection of MCs and physical domain. On each PC, the cover function is constructed to represent the local physical property. Then, the manifold elements (MEs) are generated through the shared region of

PCs. Accordingly, the NMM approximation on each ME is obtained by pasting the cover functions using the associated weight functions. More details about the above process can be found in the previous work [Zhang et al. (2010)].

For the present problem, the temperature in any ME e is approximately expressed as

$$T^h(\mathbf{x},t)=\sum_{i=1}^{n_t} w_i(\mathbf{x})T_i(\mathbf{x},t) \tag{5}$$

where n_t is the amount of PCs shared by e. $w_i(\mathbf{x})$ is the PU weight function defined on the MC containing the ith PC. $T_i(\mathbf{x},t)$ is the cover functions defined on the ith PC. For 2D continuous problems, $T_i(\mathbf{x},t)$ is frequently chosen as

$$T_i(\mathbf{x},t)=\mathbf{P}(\mathbf{x})\mathbf{a}_i(\mathbf{x},t) \tag{6}$$

where $\mathbf{a}_i$ is the thermal degrees of freedom (DOFs) defined on the ith PC. $\mathbf{P}(\mathbf{x})$ is the polynomial basis being

$$\mathbf{P}(\mathbf{x})=\begin{bmatrix}1 & x & y & \cdots\end{bmatrix} \tag{7}$$

NMM Discrete equations

The NMM discrete equations can be derived using the weighted residual method in Galerkin form [Lin (2003)]. Let $T\in H^1(\Omega)$ be the temperature trial function and $\delta T\in H^1(\Omega)$ be the corresponding test function with H^1 the first Hilbert space and δ the first order variation. A weak form of the discrete problem on a ME e is to find T^h in the finite dimensional subspace $V^h\in H^1(\Omega)$, $\forall\,\delta T^h\in V^h$ so that

$$\int_{\Omega^e}\left(\rho c\frac{\partial T}{\partial t}\delta \mathrm{T}^h+\mathbf{q}(T^h)k\cdot\mathbf{q}(\delta\mathrm{T}^h)\right)d\Omega+\lambda_T\int_{\Gamma_T^e}(T^h-\overline{T})\delta T^h d\Gamma-\int_{\Gamma_q^e}\overline{q}\delta T^h d\Gamma=0 \tag{8}$$

where λ_T is the penalty numbers adopted to enforce the essential boundary conditions due to the inconsistence of MC system with the physical boundary. Ω^e and Γ_m^e (m denotes T and q) are, respectively, the domain and/or boundary occupied or shared by the ME e.

Through Eq. (5), the test functions δT^h is expressed as

$$\delta T^h(\mathbf{x},t)=\sum_{i=1}^{n_t} w_i(\mathbf{x})\delta T_i(\mathbf{x},t) \tag{9}$$

On substituting Eqs. (5) and (9) into Eq. (8) and considering the arbitrariness of variation of DOFs, the NMM discrete equations for transient thermal conduction problems are derived as

$$\mathbf{K}_T\mathbf{T}+\mathbf{C}_T\dot{\mathbf{T}}=\mathbf{F}_T \tag{10}$$

where $\mathbf{T}$ and $\dot{\mathbf{T}}$ are, respectively, the vector of thermal DOFs and their time derivatives. $\mathbf{K}_T$, $\mathbf{C}_T$ and $\mathbf{F}_T$ are, respectively, the thermal conductivity matrix, the heat capacity matrix and the equivalent thermal load vector as

$$\mathbf{K}_T=\int_{\Omega^e}\mathbf{B}_T^{\mathrm{T}}k\mathbf{B}_T d\Omega+\lambda_T\int_{\Gamma_T^e}\mathbf{N}_T^{\mathrm{T}}\mathbf{N}_T d\Gamma \tag{11}$$

$$\mathbf{C}_T=\int_{\Omega^e}\mathbf{N}_T^{\mathrm{T}}\rho c\mathbf{N}_T d\Omega \tag{12}$$

$$\mathbf{F}_T=\lambda_T\int_{\Gamma_T^e}\mathbf{N}_T^{\mathrm{T}}\overline{T}d\Gamma-\int_{\Gamma_q^e}\mathbf{N}_T^{\mathrm{T}}\overline{q}d\Gamma \tag{13}$$

where the superscript T denotes the matrix transpose. The entries of $\mathbf{N}_T$ and $\mathbf{B}_T$ are

$$\mathbf{N}_T=\begin{bmatrix}\mathbf{N}_T^1 & \mathbf{N}_T^2 & \ldots & \mathbf{N}_T^i & \ldots & \mathbf{N}_T^{n_t}\end{bmatrix} \tag{14}$$

$$\mathbf{B}_T = \begin{bmatrix} \mathbf{B}_T^1 & \mathbf{B}_T^2 & \ldots & \mathbf{B}_T^i & \ldots & \mathbf{B}_T^{n_t} \end{bmatrix} \tag{15}$$

with

$$\mathbf{N}_T^i = \begin{bmatrix} w_i \mathbf{P} \end{bmatrix} \tag{16}$$

$$\mathbf{B}_T^i = \begin{bmatrix} (w_i \mathbf{P})_{,x} \\ (w_i \mathbf{P})_{,y} \end{bmatrix} \tag{17}$$

Numerical integration

Although the shape of MCs is user-defined, the highly developed elements in the finite element method are widely chosen. In view that the MC system can be independent of the physical domain, in this work, square elements are adopted. Further, for simplicity, the polynomial basis in Eq. (7) is set to be constant. In addition, since the shape of MEs may be diversified due to the inconsistence of MCs and physical boundary, to conveniently and accurately calculate the corresponding spatial integration in Eqs. (11) and (12), each non-triangular ME is firstly partitioned into several sub-triangles, and then the 3-point Gaussian quadrature rules are applied on each sub-triangle, the corresponding result on which finally adds up to the integration of the ME. As for the time integration, the widely used Euler backward difference method [Cebeci (2002)] is used.

Numerical examples

In this section, to verify the accuracy of the proposed method, unsteady heat conduction in an isotropic square plate is considered.

As shown Fig. 1a, the side length of the plate is L. The associated boundary and initial conditions are prescribed as

$$T(0, y, t) = T(x, 0, t) = T(L, y, t) = T(x, L, t) = 0.0 \tag{18}$$

$$T(x, y, 0) = 10\sin(x)\sin(y) \tag{19}$$

When modeling, corresponding parameters are set as: $L = \pi$, $\rho = 1.0$, $c = 1.0$ and $k = 1.0$. Accordingly, the theoretical temperature solution to this problem is [Li et al. (2011)]

$$T(x, y, t) = 10\sin(x)\sin(y)\exp(-2t) \tag{20}$$

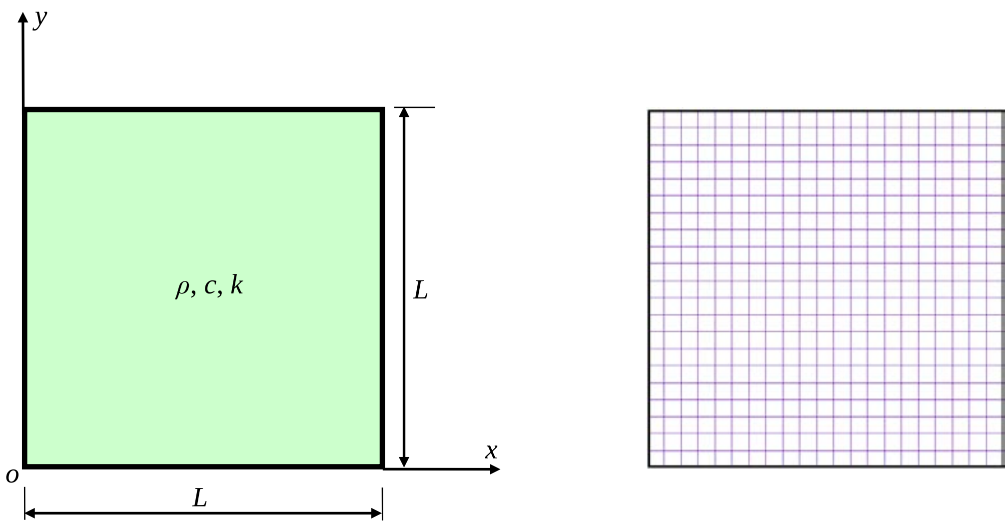

Figure 1. Transient heat conduction in a square plate: (a) physical domain and (b) discretization when *h*=0.15

In the simulation, mathematical cover system of element size (defined as the edge length of the square mathematical elements) h =0.15 is used to cover the whole plate and the associated discretized domain is illustrated in Fig. 1b, which contains 484 PCs and 441MEs. As for the time step, three values, i.e., $\Delta t = 0.1, 0.05$ and 0.02, are examined. The penalty number λ_T in Eq. (8) is taken as 1.0×10^6. The computed temperatures of two sample points A: ($\pi/4$, $\pi/4$) and B: ($\pi/2$, $\pi/2$) at different instants by the present method are, respectively, plotted in Fig. 2 and 3. For comparison, the exact results from Eq. (20) are also provided therein. Obviously, the temperatures at all time steps match well with the exact solution; what's more, with the decrease of time step, our results are getting closer to the analytical ones, which conforms to the convergence rule of the backward difference method.

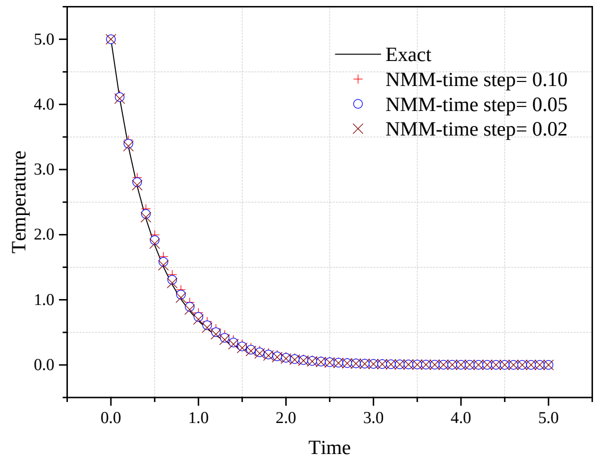

Figure 2. Computed temperatures of point A at different instants

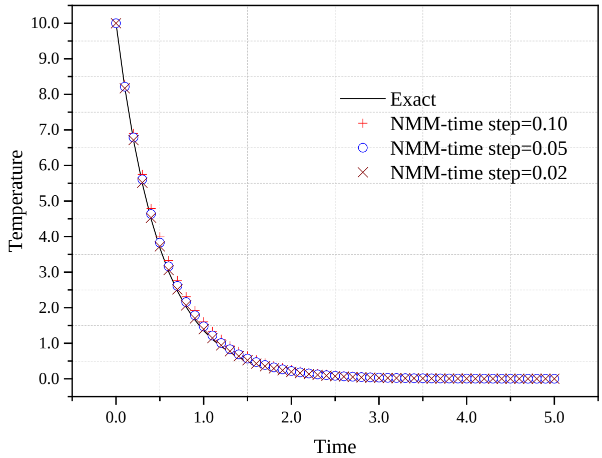

Figure 3. Computed temperatures of point B at different instants

Conclusions

In this work, the numerical manifold method has been developed to study 2D unsteady heat conduction problems. The NMM discrete equations are derived and the numerical integration schemes in the spatial and time domain are presented. A typical example is conducted to validate the proposed method. Mathematical covers formed by square elements are adopted for numerical modeling due to the inconsistence of the mathematical cover system and physical boundaries. It's found that the accuracy of the present method is satisfactory compared with the reference solutions.

Acknowledgements

The present work was supported by the National Natural Science Foundation of China (Grant No. 11462014), the Provincial Natural Science Foundation of Jiangxi, China (Grant No. 20151BAB202003) and the Science and Technology Program of Educational Committee of Jiangxi Province of China (Grant No. GJJ14526 and GJJ150752).

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