Numerical instability of staggered electromagnetic and structural coupled analysis using time integration method with numerical damping

†*T. Niho[1], T. Horie[1], J. Uefuji[1] and D. Ishihara[1]

[1]Department of Mechanical Information Science and Technology, Kyushu Institute of Technology, Japan

*Presenting author: niho@mse.kyutech.ac.jp
†Corresponding author: niho@mse.kyutech.ac.jp

Abstract

In electromagnetic and structural coupled problems such as magnetic damping vibration, the staggered method is used for coupled analyses because of its low computational cost. However, numerical instability may occur as a result of the time lag in coupled effect evaluation even if the time integration method for each phenomenon is unconditionally stable.

In this study, the stability of staggered coupled analyses is evaluated based on the spectral radius, and the stable regions of time increments with the intensity of the coupling effect are obtained. The numerical stability of the coupled analysis methods is compared for various coupling effect intensities based on the stable region.

The coupled analysis method with the conventional serial staggered algorithm and generalized–α method is most stable. The stability of the conventional parallel staggered algorithm is much improved if the generalized–α method is used.

Keywords: Numerical instability, Electromagnetic and structural coupled analysis, Coupled algorithm, Time integration method, Numerical damping.

Introduction

The use of coupled finite element analyses such as fluid–structure interaction analysis and electromagnetic–structural coupled analysis is increasing in the design of mechanical components. Coupled finite element analysis methods are classified as simultaneous (or monolithic) and staggered (or partitioned) methods. In simultaneous methods, the coupled finite element equations are obtained by combining each finite element equation for multi-physics phenomena and then solved. However, high computational cost is incurred because the matrix size becomes large. In staggered methods, multiple finite element equations are solved separately. Because the computational cost of the staggered method is low, this method is used in many coupled analyses. However, numerical instability may occur owing to time lag in coupled effect evaluation even if the time integration method for each phenomenon is unconditionally stable.

Many studies of staggered methods have been performed for fluid–structure interaction problems. In addition to the conventional serial staggered (CSS) algorithm, which is widely used for staggered analysis, several coupled algorithms have been proposed such as the conventional parallel staggered (CPS) algorithm, improved serial staggered algorithm and improved parallel staggered algorithm; and then the numerical stability, result accuracy and computing time of these methods have been discussed[1].

Magnetic damping vibration is one type of electromagnetic and structural coupled problem. Studies have focused on magnetic damping vibration analysis, which is required for the design of conductive structures located in a strong magnetic field, such as those in future fusion reactors or magnetically levitated vehicles. Several coupled analysis methods have been compared for magnetic damping vibration with the bending mode[2] and with the bending and torsional mode[3] from the viewpoint of the modeling, formulation, type of element, and time integration method. In the past few years, the geometrical nonlinearity of magnetic damping vibration has been discussed[4], and a coupled analysis method using a Lagrangian approach has been proposed[5]. However, numerical instability occurs in magnetic damping vibration analysis even if unconditionally stable time integration methods are used.

In this study, a stability evaluation method is proposed for the coupled finite element analysis of magnetic damping vibration. In this method, the stability is evaluated by the spectral radius obtained from the coupled eigenmode and the time integration scheme. Next, the numerical stability is examined by the stable region for various coupled analysis methods that are combined with a coupled algorithm and a time integration method with numerical damping.

Coupled Finite Element Analysis Method for Magnetic Damping Vibration Problem

Magnetic Damping Vibration

Magnetic damping vibration occurs in a conductive structure located in a magnetic field. A conductive structure is vibrated by the Lorentz force which is induced by an eddy current and a magnetic field. While the structure is vibrating, the electromotive force reduces the eddy current and vibration.

Finite Element Equations

The T method is used for eddy current analysis of the magnetic damping vibration problem of a thin shell structure[6]. The matrix equation of the eddy current analysis is expressed using the nodal point normal component $\boldsymbol{T}$ of the current vector potential and nodal point deformation vector $\mathbf{u}$:

$$\mathbf{U}\dot{\boldsymbol{T}} + \mathbf{R}\boldsymbol{T} = \mathbf{C_e}\dot{\mathbf{u}} + \dot{\boldsymbol{B}}^{ex}. \tag{1}$$

Here, $\mathbf{U}$, $\mathbf{R}$, $\mathbf{C_e}$, and $\dot{\boldsymbol{B}}^{ex}$ are the inductance matrix, the resistance matrix, the coupling sub-matrix of electromotive force, and the time-varying external magnetic field, respectively.

The matrix equation of the structural analysis is expressed by

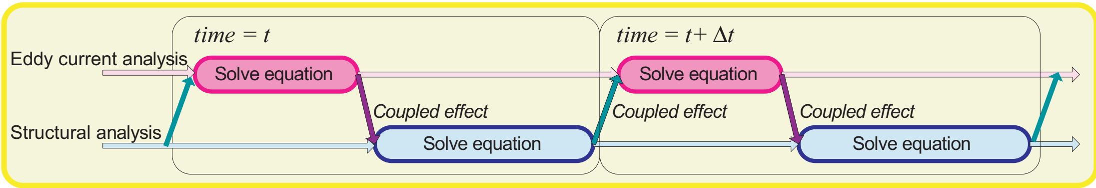
$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{C_s}\boldsymbol{T} + \boldsymbol{F}^{ex}, \tag{2}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C_s}$, and $\boldsymbol{F}^{ex}$ are the mass matrix, the stiffness matrix, the coupling sub-matrix of the Lorentz force, and the external force, respectively.

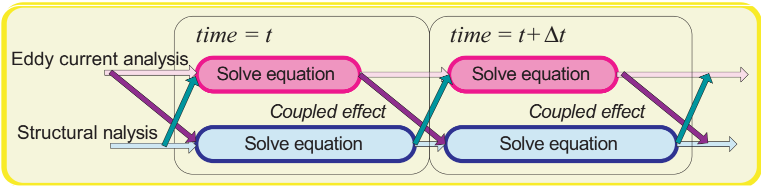
Coupled Algorithms

The coupled analysis methods for magnetic damping vibration are classified as simultaneous and staggered methods. In the simultaneous method, the coupled finite element equation obtained by combining Eqs. (1) and (2) has been solved[6] and shown to be unconditionally stable[7]. In the staggered method, Eqs. (1) and (2) are solved separately and alternately. However, it is conditionally stable even if unconditionally stable time integration methods are used for each equation because the solution diverges by numerical instability under specific conditions, for example, according to the intensity of the magnetic field and the time increment. In addition to the CSS algorithm, the CPS algorithm have been proposed for fluid–structure interaction analysis[1]. According to the previous studies, the CPS algorithm has weak stability. In this study, the numerical stability of these coupled algorithm are discussed for staggered methods of magnetic damping vibration analysis.

Fig. 1 shows the data flow between the eddy current analysis and the structural analysis using the CSS and CPS algorithms for magnetic damping vibration analysis. In the CSS algorithm,



(a) Conventional serial staggered (CSS) algorithm



(b) Conventional parallel staggered (CPS) algorithm

Figure 1: Procedures of staggered coupled algorithms for eddy current and structural coupled analyses.

Eq. (1) for the eddy current analysis is solved using the results from the previous time step of the structural analysis to evaluate the coupling term in Eq. (1). Then, Eq. (2) for the structural analysis is solved using the results of eddy current analysis to evaluate the coupling term in Eq. (2). In the CPS algorithm, Eq. (1) for the eddy current analysis and Eq. (2) for the structural analysis are solved simultaneously and separately in each time step. The terms for the coupled effect in Eqs. (1) and (2) are evaluated using the results from the previous time step.

Coupld Analysis Methods

For eddy current analysis, the backward difference method is applied. Eq. (1) becomes

$$(\mathbf{U} + \Delta t\mathbf{R})\,\boldsymbol{T}_{t+\Delta t} = \Delta t\mathbf{C}_e\dot{\mathbf{u}}_{t+\Delta t} + \mathbf{U}\boldsymbol{T}_t + \Delta t\dot{\boldsymbol{B}}_t^{ex}. \tag{3}$$

The backward difference method is unconditionally stable for uncoupled eddy current analysis.

For structural analysis, two types of time integration methods are applied. By using the parameter ρ_∞ to control the numerical dissipation, Eq. (2) becomes

$$\begin{aligned} & \left\{(1-\alpha_m)\frac{1}{\beta\Delta t^2}\mathbf{M} + (1-\alpha_f)\,\mathbf{K}\right\}\mathbf{u}_{t+\Delta t} \\ = \;& (1-\alpha_f)\,\boldsymbol{F}^{ex}_{t+\Delta t} + \alpha_f\boldsymbol{F}^{ex}_t \\ + \;& \mathbf{C}_s\left\{(1-\alpha_f)\,\boldsymbol{T}_{t+\Delta t} + \alpha_f\boldsymbol{T}_t\right\} \\ - \;& \mathbf{M}\left[(1-\alpha_m)\left\{\left(1-\frac{1}{2\beta}\right)\ddot{\mathbf{u}}_t - \frac{1}{\beta\Delta t}\dot{\mathbf{u}}_t - \frac{1}{\beta\Delta t^2}\mathbf{u}_t\right\} + \alpha_m\ddot{\mathbf{u}}_t\right] \\ - \;& \alpha_f\mathbf{K}\mathbf{u}_t, \end{aligned} \tag{4}$$

where

$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1},\ \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1},\ \delta = \frac{1}{2} - \alpha_m + \alpha_f,\ \beta = \frac{(1-\alpha_m+\alpha_f)^2}{4}$$

for Newmark's β method ($\delta = 1/2,\ \beta = 1/4$) with $\alpha_m = \alpha_f = 0$, and the asymptotic annihilation case $\rho_\infty = 0$ of the generalized–α method ($\delta = 3/2,\ \beta = 1$)[8]. These time integration methods are unconditionally stable for uncoupled structural analysis.

For the electromotive force or the coupled effect in eddy current analysis, the coupling term $\mathbf{C}_\mathrm{e}\dot{\boldsymbol{u}}$ is evaluated under the assumption that $\dot{\boldsymbol{u}}$ is equal to $\dot{\boldsymbol{u}}_t$ for both coupled algorithms. The Lorentz force or the coupled effect in structural analysis is evaluated in a different way for each coupled algorithm. In the CSS algorithm, the coupling term for structural analysis $\mathbf{C}_s\boldsymbol{T}$ can be evaluated using $\boldsymbol{T}_{t+\Delta t}$ obtained from the eddy current analysis in the same time step. In the CPS algorithm, $\boldsymbol{T}$ is assumed to be $\boldsymbol{T}_t$ to evaluate the coupling term.

Stability Analysis Method of Magnetic Damping Vibration Analysis

The stability of a time integration method for uncoupled analysis can be generally evaluated using the spectral radius[9]. The stability of magnetic damping vibration analysis, which is one type of coupled analysis, is also evaluated using the spectral radius[10].

The stability analysis method for the combination of the vibration mode m and the eddy current mode n is described below. By ignoring the term of the external transient magnetic field in Eq. (3) for eddy current analysis and using the mode amplitude factor $\bar{u}^{(m)}$ for the vibration mode m, the mode amplitude factor $\bar{T}^{(n)}$ for the eddy current mode n is expressed as

$$\left(\bar{U}^{(n)} + \Delta t\bar{R}^{(n)}\right)\bar{T}^{(n)}_{t+\Delta t} = \Delta t\bar{C}^{(m)(n)}_e\dot{\bar{u}}^{(m)}_{t+\Delta t} + \bar{U}^{(n)}\bar{T}^{(n)}_t, \tag{5}$$

where $\bar{U}^{(n)}$ and $\bar{R}^{(n)}$ are respectively the modal inductance and modal resistance of eddy current mode n, and $\bar{C}^{(m)(n)}_e$ is the modal electromotive force of the coupling effect between vibration mode m and eddy current mode n. On the other hand, by ignoring the term of the external force in Eq. (4) for structural analysis, $\bar{u}^{(m)}$ is expressed as

$$\begin{aligned} & \left\{(1-\alpha_m)\frac{1}{\beta\Delta t^2}\bar{M}^{(m)} + (1-\alpha_f)\bar{K}^{(m)}\right\}\bar{u}^{(m)}_{t+\Delta t} \\ = \; & \bar{C}^{(m)(n)}_s\left\{(1-\alpha_f)\bar{T}^{(n)}_{t+\Delta t} + \alpha_f\bar{T}^{(n)}_t\right\} \\ - \; & \bar{M}^{(m)}\left[(1-\alpha_m)\left\{\left(1-\frac{1}{2\beta}\right)\ddot{\bar{u}}^{(m)}_t - \frac{1}{\beta\Delta t}\dot{\bar{u}}^{(m)}_t - \frac{1}{\beta\Delta t^2}\bar{u}^{(m)}_t\right\} + \alpha_m\ddot{\bar{u}}^{(m)}_t\right] \\ - \; & \alpha_f\bar{K}^{(m)}\bar{u}^{(m)}_t, \end{aligned} \tag{6}$$

where $\bar{M}^{(m)}$ and $\bar{K}^{(m)}$ are respectively the modal mass and the modal stiffness for vibration mode m, and $\bar{C}^{(m)(n)}_s$ is the modal Lorentz force for the coupled effect between vibration mode m and eddy current mode n.

By combining Eqs. (5) and (6) and moving terms according to the time, the recurrence equation of the magnetic damping vibration analysis becomes

$$\left\{\ddot{\bar{u}}^{(m)}_{t+\Delta t}\;\dot{\bar{u}}^{(m)}_{t+\Delta t}\;\bar{u}^{(m)}_{t+\Delta t}\;\bar{T}^{(n)}_{t+\Delta t}\right\}^T = \mathbf{A}\left\{\ddot{\bar{u}}^{(m)}_t\;\dot{\bar{u}}^{(m)}_t\;\bar{u}^{(m)}_t\;\bar{T}^{(n)}_t\right\}^T. \tag{7}$$

The stability of the magnetic damping vibration analysis can be evaluated using the modulus of the complex eigenvalue $|\lambda^{(m)(n)}|$, which is the spectral radius of the amplitude matrix $\mathbf{A}$. If any $|\lambda^{(m)(n)}|$ is greater than 1.0, the coupled analysis method is considered unstable.

The stability analysis method is applied to the coupled analysis method with CSS algorithm and

Newmark's β method. The eigenvalues λ of $\mathbf{A}$ are obtained from the characteristic equation

$$\lambda^4 + \left\{ -2 + \frac{4\omega^2 \Delta t^2}{4+\omega^2 \Delta t^2} - \frac{1}{1+\phi \Delta t} - \frac{2\Delta t^2}{4+\omega^2 \Delta t^2} \frac{1}{1+\phi \Delta t} \frac{\bar{C}_e \bar{C}_s}{\bar{M}\bar{U}} \right\} \lambda^3$$
$$+ \left\{ 1 + \left(2 - \frac{4\omega^2 \Delta t^2}{4+\omega^2 \Delta t^2} \right) \frac{1}{1+\phi \Delta t} \right\} \lambda^2$$
$$+ \left\{ -\frac{1}{1+\phi \Delta t} + \frac{2\Delta t^2}{4+\omega^2 \Delta t^2} \frac{1}{1+\phi \Delta t} \frac{\bar{C}_e \bar{C}_s}{\bar{M}\bar{U}} \right\} \lambda = 0, \tag{8}$$

where

$$\omega = \sqrt{\frac{\bar{K}}{\bar{M}}}, \quad \phi = \frac{\bar{R}}{\bar{U}},$$

and the superscripts $^{(m)}$ and $^{(n)}$ of the modal coefficients are omitted. The characteristic equations for other coupled analysis methods are obtained in the same way as described above.

The values of ω, ϕ, and $\frac{\bar{C}_e \bar{C}_s}{\bar{M}\bar{U}}$ in Eq. (8) depend on the material properties, geometric configuration, and intensity of the coupled effect, so they are obtained using theoretical and finite element solutions. The value of ω is obtained from Young's modulus, mass density, length, width, and thickness of the plate by using the theoretical solution for a thin flexible plate. For the values of ϕ and $\frac{\bar{C}_e \bar{C}_s}{\bar{M}\bar{U}}$, the characteristic equation of the magnetic damping vibration[11] is used. By combining modal Eqs. (1) and (2), the characteristic equation becomes

$$\alpha_c^3 + \phi\, \alpha_c^2 + \left(\omega^2 - \frac{\bar{C}_e \bar{C}_s}{\bar{M}\bar{U}} \right) \alpha_c + \omega^2 \phi = 0, \tag{9}$$

where α_c is coupled eigenvalue that depends on the geometry. By using the result of the eigenvalue of the coupled finite element monolithic matrix equation[11] combined with Eqs. (1) and (2), Eq. (9) becomes a complex linear equation with unknown variables ϕ and $\frac{\bar{C}_e \bar{C}_s}{\bar{M}\bar{U}}$, which are determined through this equation. Therefore, the eigenvalue of the characteristic equation Eq. (8) can be solved numerically for each Δt using the Newton method, and the stability can be evaluated using the spectral radius $|\lambda|$.

Results of stability analysis of coupled analysis methods

Magnetic Damping Vibration of Elastic Plate

The coupled analyses are performed for a magnetic damping vibration problem, as shown in Fig. 2[2]. A copper rectangular plate clamped at one end is placed in a longitudinal steady magnetic field B_x and transient magnetic field

$$B_z = 5.5 \times 10^{-2} \exp \frac{-t}{6.6 \times 10^{-3}} \text{ [T]}, \tag{10}$$

that is applied perpendicularly to the plate surface. The Lorentz force produced by both the eddy current induced by B_z and B_x causes bending vibration. While the plate is vibrating, the electromotive force induced by the vibration velocity and B_x induces a coupling effect to reduce the eddy current and vibration.

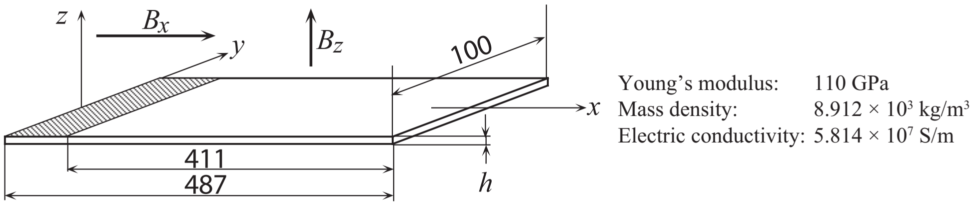


Figure 2: Schematic diagram of a clamped plate placed in electromagnetic field.

Verification of the stability analysis method

The spectral radii $|\lambda|$ obtained by the stability analysis for the coupled analysis methods using the CSS algorithm are shown in Fig. 3 for $B_x = 0.5$ T. The time increment Δt is normalized by the natural period $\tau_0 = 9.37 \times 10^{-2}$ s for the first vibration mode. The critical time increment $\Delta t_c^{(s)}$ is defined from the limit of Δt when all $|\lambda|$ values become less than or equal to 1.0. If any value of $|\lambda|$ is greater than 1.0, the coupled solution is unstable. For coupled analysis method with generalized–α method, $|\lambda|$ is always less than 1.0.

The validity of the stability analysis method should be confirmed using the coupled finite element analysis for various values of Δt, in which the staggered method for both the vibration mode response analysis and the eddy current mode response analysis is used. For coupled analysis methods with Newmark's β method, coupled finite element analyses are performed under the time increment conditions of both $\Delta t < \Delta t_c^{(s)}$ and $\Delta t > \Delta t_c^{(s)}$. For coupled analysis method with the generalized–α method, coupled finite element analysis is performed using large Δt, such as the natural period τ_0.

Fig. 4 shows the deflections at the free end of the plate. According to Fig. 4(a), the results obtained using the method with Newmark's β method is stable when $\Delta t < \Delta t_c^{(s)}$, but it is unstable when $\Delta t > \Delta t_c^{(s)}$. For the method with the generalized–α method, instability is not observed in Fig. 4(b) even if Δt is set to be as large as the natural period.

The $|\lambda|$ values obtained by the stability analysis are shown in Fig. 5 for the coupled analysis methods with the CPS algorithm. For the method with the generalized–α method, $|\lambda|$ is always less than 1.0. Fig. 6 shows the deflections of the plate obtained using the CPS algorithm. According to Fig. 6(a), the results obtained using the method with Newmark's β method is stable when $\Delta t < \Delta t_c^{(s)}$, but it is unstable when $\Delta t > \Delta t_c^{(s)}$. For the method with the generalized–α method, instability is not observed in Fig. 6(b) even if Δt is set to be as large as the natural period. Therefore, the validity of the stability evaluation method using the spectral radius is confirmed for the coupled analysis methods for the magnetic damping vibration.

Comparison of Numerical Stability

The numerical stability of the coupled analysis methods is compared for various intensities of the coupling effect. Fig. 7 shows the normalized critical time increment $\Delta t_c^{(s)}/\tau_0$ for various steady magnetic fields B_x, which is proportional to the intensity of the coupling effect. The lower left region of each curve is stable, whereas the upper right region is unstable because of the high intensity of the coupling effect. Although Newmark's β method, the generalized–α method and the backward difference method are unconditionally stable for uncoupled analysis, all coupled analysis methods using these time integration methods are conditionally stable on account of the staggered coupled analysis method. Because $\Delta t_c^{(s)}/\tau_0$ becomes smaller with increasing intensity of the coupling effect, the stability deteriorates with the coupling effect. The stable regions of the coupled analysis methods with the generalized–α method are larger than those with Newmark's β method. This is because the numerical damping of the generalized–α method may suppress the instability induced by these coupled analysis methods.

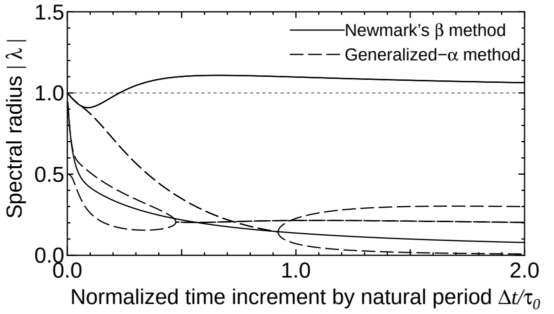
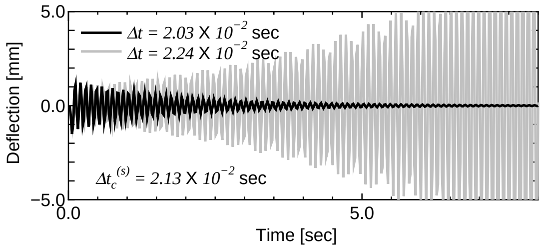
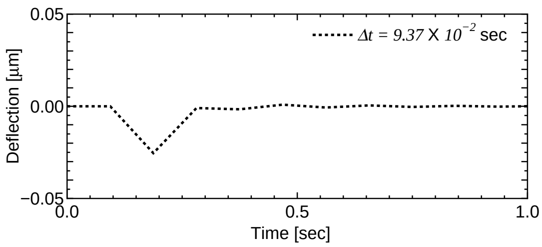


Figure 3: Results of spectral radii $|\lambda|$ of coupled analysis methods with CSS algorithm.



(a) Newmark's β method



(b) Generalized–α method

Figure 4: Deflection of the plate obtained by coupled analysis methods with CSS algorithm.

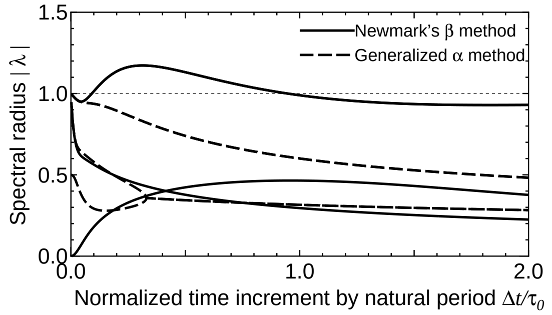
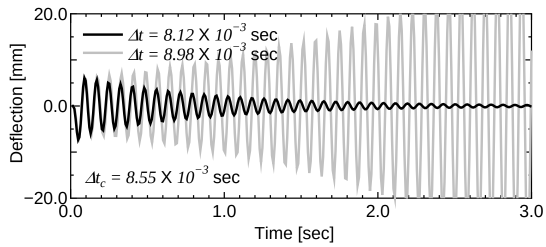
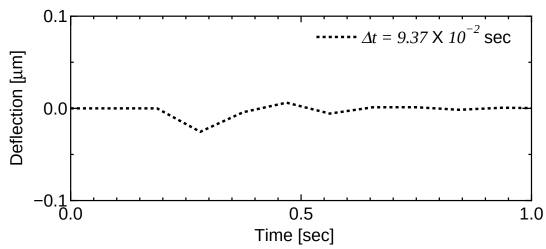


Figure 5: Results of spectral radii $|\lambda|$ of coupled analysis methods with CPS algorithm.



(a) Newmark's β method



(b) Generalized–α method

Figure 6: Deflection of the plate obtained by coupled analysis methods with CPS algorithm.

When results for the CPS algorithm are compared with those for the CSS algorithm, the stable regions of the CPS algorithm are smaller than those of the CSS algorithm, which is the same tendency as in the fluid–structure interaction analysis with the CPS algorithm[1]. This may be because the time lag of coupled effect evaluation for the CSS algorithm treats only the electromotive force, whereas that for the CPS algorithm treats both the electromotive force and the Lorentz force. Although the stability of the CPS algorithm was worse than that of the CSS algorithm in general, it was much improved when using the generalized–α method, and this offers the advantage of a shorter computing time.

Conclusions

A stability evaluation method using the spectral radius was proposed and applied to the coupled finite element analysis of magnetic damping vibration. The stability was evaluated for coupled analysis methods that were combined with a coupled algorithm and time integration method. The validity of the stability evaluation method and the results of stability analysis were confirmed through comparisons with the results of coupled finite element analyses.

The coupled analysis method with the CSS algorithm and generalized–α method is the most

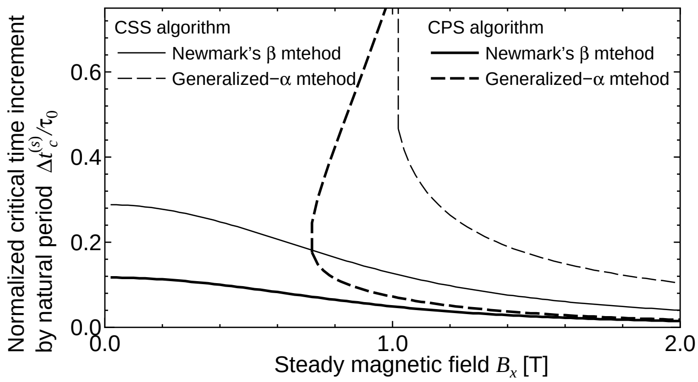


Figure 7: Stability limit of coupled analysis methods as a function of intensity of the coupled effect. Lower left region of the curve is stable region, and upper right region of the curve is unstable region.

suitable for the coupled finite element analysis of the magnetic damping problem. The generalized–α method is superior to Newmark's β method from the viewpoint of the stability of the coupled analysis. The CPS algorithm is considered inferior to the CSS algorithm in terms of numerical stability, but the stability is much improved if the generalized–α method is used. Then, the advantage of parallel computing can be better utilized when the intensity of the coupling effect is low.

References

[1] Farhat, C. and Lesoinne, M. (2000) Two efficient staggered algorithms for the serial and parallel solution of three–dimensional nonlinear transient aeroelastic problems, *Computer Methods in Applied Mechanics and Engineering* **182**, 499–515.

[2] Turner, L. R. and Hua, T. Q. (1990) Results for the cantilever beam moving in crossed magnetic fields, *COMPEL* **9**, 205–216.

[3] Takagi, T. (1995) Summary of the results for magnetic damping in torsional mode (TEAM problem 16), *COMPEL* **14**, 77–89.

[4] Zhang, J., Yan, Z., Ding, Q., Wu, H. and Pan, M. (2013) Analysis of magnetoelastic interaction of cantilever conductive thin plate with nonlinear dynamic response, *European Journal of Mechanics. A/Solids* **37**, 132–138.

[5] Li, W., Dong, H., Yuan, Z. and Chen, Z. (2014) Numerical analysis of electromagneto-mechanical coupling using Lagrangian approach and adaptive time stepping method, *International Journal of Applied Mechanics* **6**, 1450051.

[6] Horie, T. and Niho, T. (1994) Electromagnetic and mechanical interaction analysis of a thin shell structure vibrating in an electromagnetic field, *International Journal of Applied Electromagnetics in Materials* **4**, 363–368.

[7] Niho, T., Horie, T. and Tanaka, Y. (2000) Numerical instability of magnetic damping problem of elastic plate *IEEE Transactions on Magnetics* **36**, 1373–1376.

[8] Chung, J. and Hulbert, G. M (1993) A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized–α method, *Journal of Applied Mechanics* **60**, 371–375.

[9] Bathe, K. J. and Wilson, E. L. (1973) Stability and accuracy of direct time integration methods, *Earthquake Engineering and Structural Dynamics* **1**, 283–291.

[10] Tanaka, Y., Horie, T., Niho, T., Shintaku, E. and Fujimoto, Y. (2004) Stability of augmented staggered method for electromagnetic and structural coupled problem, *IEEE Transactions on Magnetics* **40**, 549–552.

[11] Horie, T., Niho, T. and Kawano, T. (1995) *Coupling intensity parameter and its dependence for magnetic damping problem*, *Electromagnetic Phenomena Applied to Technology, Enokizono, M. and Todaka, T. eds.*, JSAEM, Tokyo, Japan, 61–68.