Minimum volume of the longitudinal fin with rectangular and triangular profile by a modified Newton-Raphson method

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Abstract

The minimum volume of nonlinear longitudinal fin with rectangular and triangular profile by using the modified Newton-Raphson method is present in this paper. The dimension of the fin profile is regarded as optimization variables. Furthermore, a mechanism called "volume updating" is added into the modified Newton-Raphson algorithm to obtain the minimum volume of the fin. Two examples are illustrated to demonstrate the proposed method. The obtained results showed that the proposed method use efficiently and accurately in finding the minimum volume of the nonlinear longitudinal fin problem with the rectangular and triangle profile.

Keywords: Shape Optimization; Modified Newton Raphson; Rectangular fin; Triangular fin.

Introduction

Fin or extended surface is used widely in various industrial applications when we want to improve the convective heat transfer from a hot surface where cooling is required [1]. The use of fins increases the volume or mass of systems and rise the costs of production. Consequently, the optimization of fins for light weight and high efficiency and compact heat exchanger system is of great interested and have been done in the past several decades.

Fin optimization problems can be divided into two approaches. The first approach of optimization problem is to select a simple profile (i.e. rectangular or triangular) and then determine the dimensions of fin so that either maximize the heat transfer rate for a given volume or minimize the volume of fin for a specified heat dissipation. In the second approach, the shape of fin is determined so that the volume of the material used is minimum for a given heat loss. The criterion for this second approach of fin optimization problems was first proposed by Schmidt [2]. For purely conduction and convection fins, the author suggested that the minimum volume of the optimization fins is a parabolic shape. Unfortunately, the parabolic profiles of optimization fin in the second approach are curved surface with zero tip thickness which is too complex and expensive to manufacture. Thus, the first approach of problems is more relevant and important than the second type of problems.

In fact, the rectangular and triangular profiles are widely used in the heat exchanger system due to the ease of fabrication. As a result, more studies have been performed to determine the optimal size of these types. Under the assumption constant thermal parameters, negligible effects of heat transfer from the tip, and approximation of one-dimensional heat transfer equation, the optimization of rectangular and triangular profiles is treated in the book by Kraus [1]. Aziz [3] published an article which present a literature survey on optimum dimension of this object. Aziz [4, 5] optimized the rectangular and triangular fins with convective boundary conditions and presented the optimum design of a rectangular fin with a step change in cross-sectional area under the constant thermal parameters. Under the variable thermal parameters, Yu [6] studied the optimization of rectangular by applying Taylor transformation method. Recently, by using the differential transformation method, Poozesh [7] presented the efficiency of convective-radiative fin with temperature-dependent thermal conductivity. In Poozesh's paper, the effects of convection-conduction parameter, thermal conductivity parameter and the radiation-conduction parameter on efficiency of fin are considered and discussed. For a bi-dimensional analysis, Kang [8, 9] estimated the optimum dimension of annular fins with rectangular profile under thermally asymmetric convective and radiating condition as well as optimized an annular trapezoidal fin using a new approach to a two-dimension analytical method. However, these above researches were performed based on the analytical method under assumption of constant thermal parameters. The drawback of analytical methods is that they cannot solve the general non-linear fin design problem. However, none of previous published papers however propose the effective methods to minimize the volume of rectangular and triangle fins for general non-linear fin design problem until now.

In this paper, a proposed an effective method is presented to find the minimum volume of longitudinal fin of rectangular and triangular profiles for general high non-linear fin design problem based on modified Newton Raphson method (MNR). A mechanism called as "volume updating" is added in MRN algorithm to obtain the minimum volume of optimum fin. The potential and feasibility of applying MNR as an optimization method on the fin problems will be demonstrated in this work.

Problem Statement

Consider a longitudinal symmetric fin model with rectangular and triangular profiles as Figure 1, in the steady state condition, the general heat transfer equation without internal heat source for the two-dimensional model given by the semi-cross-section of the natural convection and radiation cooled fin takes on the following forms:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=0 \text{ in domain of fin} \quad (1)$$

$$-kA_b\frac{\partial T}{\partial x}=q_{flow} \text{ at fin base} \quad (2)$$

$$-k\left(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}\right).\mathbf{n}=h(T-T_\infty)+\varepsilon\sigma(\mathrm{T}^4\text{-}\mathrm{T}_{sur}^4) \text{ at convective surface} \quad (3)$$

$$k\frac{\partial T}{\partial y}=0 \text{ at symmetric of fin} \quad (4)$$

where T is the unknown temperature field over the cross-section domain of fin, k is the heat thermal conductivity, A_b is the fin cross section area at the base, q_{flow} is the inward total heat loss at the base, h is the convective heat transfer coefficient, ε is emissivity coefficient, σ is Estefan-Boltzmann constant, T_∞ and T_{sur} is the ambient and surrounding temperature

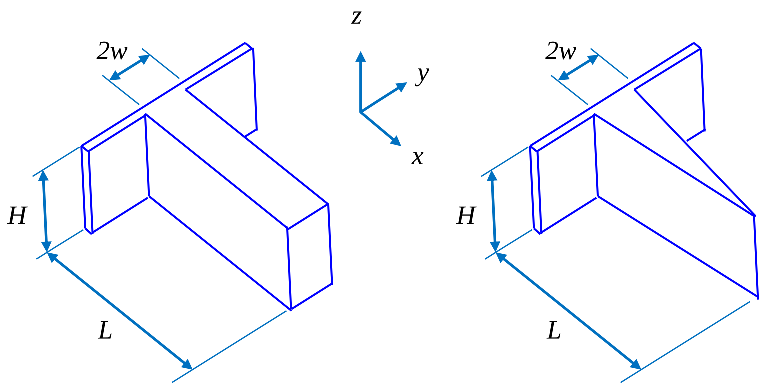

Figure 1. The longitudinal fin with rectangular and triangular shape

respectively, and n is the exterior normal vector of the convective surface. In general, the coefficients k, h, ε are constant or functions of temperature.

When the shape of fin and all boundary condition is known and given, the temperature field of fin and the base temperature could be estimated by solving the non-linear fin design problem (Eqs.(1-4)). This direct problem is solved by the finite element method (FEM) [10].

The Optimization Problem

Modified Newton Raphson

In this paper, the purpose of optimization process is to minimize the volume of the longitudinal fins of rectangular and triangular profiles for a given heat loss and the specified base temperature. Therefore, the dimensions of fin profiles are regarded as optimization variables. For this two-dimensional geometric problem, the dimension of fin is determined by the length and width of fin. We will thus have 2 optimization variables for both of rectangular and triangle profiles as shown in Figure 2. Besides, to remain the continuity during the optimization process, the position of control points must satisfy the following conditions:

$$\begin{cases} x > 0 \\ y > 0 \end{cases} \tag{5}$$

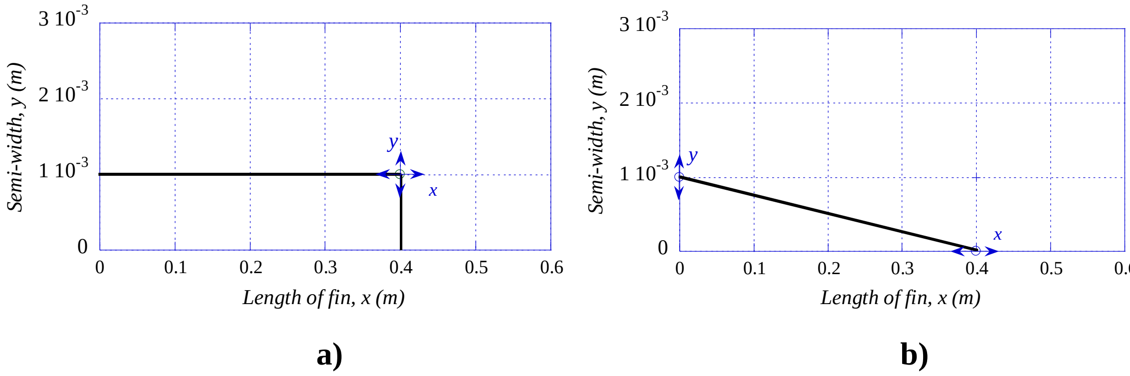

Figure 2. The profile of fins and their optimization variables: a) rectangular fin; b) triangular fin

MNR method [11] is used to find out the minimum volume by finding the optimal position of control points. The proposed method directly formulates the problem from two comparisons between the calculated and the expected temperature at the base, and between the calculated and expected volume of the fin. Therefore, the expected base temperature T_x^i and the expected fin volume V_x are necessary to be given first; the calculated temperature T_c^i and the calculated fin volume V_c are evaluated from direct problem. Then, the estimation of optimal fin shape can be recast as the solution of a set of nonlinear equations as following:

$$\begin{cases} T_c^i - T_x^i = 0 \\ V_c - V_x = 0 \end{cases} \quad i = 1, 2 \ldots M \tag{6}$$

where, M is the number of the temperature equation which is obtained from the base. As a result, there are M+1 equations in Eq. (6).

The characteristic of fin is that the fin width compared to the fin length is very small. Therefore, the variation of the base temperature along with the width of fin could be neglected. Consequently, the expected temperature at the base would be assigned to one expected value. Furthermore, since the value of fin volume is very small compared to the base temperature, the volume value is converted into the temperature value so that the influence of fin volume and base temperature in Eq. (6) is the same. Subsequently, Eq. (6) can be re-written as following:

$$\begin{cases} T_c^i - T_x = 0 \\ \hat{V}_c - T_x = 0 \end{cases} \tag{7}$$

where, T_{xpcd} is the expected temperature at the base and $\hat{V}_c$ is the converted volume given by:

$$\hat{V}_c = \frac{V_c}{V_x}.T_x \tag{8}$$

The detail procedure to solve Eq. (7) can be shown as following:

$$\mathbf{T} = [\{T_c^1 - \mathrm{T}_x\}, \{T_c^2 - \mathrm{T}_x\}, \cdots, \{T_c^M - \mathrm{T}_x\}, \{\hat{V}_c - \mathrm{T}_x\}]^T = \{\hat{T}_c\}^T \tag{9}$$

where, $\hat{T}_c$ is the component of vector $\mathbf{T}$.

The optimization variables are set as following:

$$\boldsymbol{\chi} = \{x, \mathrm{y}\}^T = \{\chi_1, \chi_2\}^T = \{\hat{\chi}_v\} \tag{10}$$

where, x and y are the dimensions of the fin (as Fig. 2), $\hat{\chi}_v$ is the component of vector $\boldsymbol{\chi}$

The derivative of $\hat{\Phi}_c$ with respect to $\hat{\chi}_v$ is can be expressed as following:

$$\mathbf{S} = \frac{\partial \hat{T}_c}{\partial \hat{\chi}_v} \tag{11}$$

where, $\mathbf{S}$ is the sensitivity matrix.

With the above derivatives from Eq. (6) to Eq. (11), we have the following equation:

$$\boldsymbol{\Delta}_k = -\left[\mathbf{S}^T(\boldsymbol{\chi}_k)\mathbf{S}(\boldsymbol{\chi}_k)\right]^{-1}\mathbf{S}^T(\boldsymbol{\chi}_k)\mathbf{T}(\boldsymbol{\chi}_k) \tag{12}$$

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \lambda \boldsymbol{\Delta}_k \tag{13}$$

where, λ is the factor to adjust the step size of $\boldsymbol{\Delta}_k$ so that the constraints of Eq. (5) are satisfied.

From Eq. (9), it is claimed that the solution can be achieved when the base temperature and the appropriate volume of fin is given. However, the minimum volume of the fin is unknown prior and is the optimization goal. To solve this problem, an approach called "volume updating" is added into the modified Newton Raphson algorithm. This approach is based on "curve fitting" mechanism of the modified Newton Raphson method. In this mechanism, the obtained solution is the best approximation which is defined as that which minimizes the sum of squared differences between the computed and expected value. As a result, in Eq. (9), the larger value of N is, the closer solution to the expected temperature compared to the expected volume is. Consequently, "volume updating" approach is performed as following:

Step 1: Set a large value for M and guess a small initial value of fin volume.
Step 2: Use Eqs. (9-13) to find the best solution.
Step 3: Update the new volume obtained from the best solution of step 2 and return to step 1.
Step 4: Terminate the process if the stopping criterion is satisfied.

The stopping criteria

The modified Newton Raphson method from Eq. (11) to Eq. (15) is used to determine the optimal location of the control points which are presented as the unknown variables, $\boldsymbol{\chi}$. The step size $\boldsymbol{\Delta}_k$ goes from $\boldsymbol{\chi}_k$ to $\boldsymbol{\chi}_{k+1}$ and it is determined from Eq. (16). Once $\boldsymbol{\Delta}_k$ is calculated, the iterative to determine $\boldsymbol{\chi}_{k+1}$ is executed until the stopping criterion is satisfied.

There are two stopping criteria used in the proposed method. One is for updating the volume and another is for modified Newton Raphson method. Base on the discrepancy principle [12], the volume would be updated when both of two criteria are satisfied as following:

$$\begin{cases} \mathbf{T}_c - T_x \geq 0 \\ \left\| \mathbf{J}\left(\chi_{k+1}\right) - \mathbf{J}\left(\chi_\kappa\right) \right\| \leq \delta \left\| \mathbf{J}\left(\chi_{k+1}\right) \right\| \end{cases} \tag{14}$$

where,

$$\left\| \mathbf{J}\left(\boldsymbol{\chi}_{k+1}\right) \right\| = \sum_{i=1}^{M} \left[\mathbf{T}_c^i - T_x \right]^2 + \left[\hat{V}_c - T_x \right]^2 \tag{15}$$

and the stopping criteria is given by

$$\left\| \mathbf{T}_c - T_x \right\| \leq e \left\| T_x \right\| \tag{16}$$

or

$$\left\| \mathbf{J}\left(\chi_{k+1}\right) - \mathbf{J}\left(\chi_\kappa\right) \right\| \leq \delta \left\| \mathbf{J}\left(\chi_{k+1}\right) \right\| \tag{17}$$

where, e and δ are small positive value known as the convergence tolerances.

Computational Algorithm

The procedure for the proposed method can be summarized as following:

Given overall convergence tolerance e and δ , the initial control point $\boldsymbol{\chi}_0$, the initial volume of fin V_x^0 , and the adjusting factor λ (say $\lambda = 1$ in the present work). The value $\boldsymbol{\chi}_k$ is known at the iteration as following:

Step 1: Solve the direct problem Eqs. (1-4), and compute $\mathbf{T}_c$.
Step 2: Integrate $\mathbf{T}_c$ with $\mathbf{T}_x$ through Eq. (9) to construct $\mathbf{T}$.
Step 3: Calculate the sensitivity matrix $\mathbf{S}$ through Eq. (11).
Step 4: Knowing $\mathbf{S}$ and $\mathbf{T}$, calculate the step size $\boldsymbol{\Delta}_k$ from Eq. (12).
Step 5: Calculate $\boldsymbol{\chi}_{k+1}$ through Eq. (13).

Step 6: If condition of Eq. (5) is not satisfied, replace $\lambda = 0.1\lambda$ and return to step 5. Otherwise, accept the new control points χ_{k+1} and set $\lambda = 1$ again.

Step 8: Update the fin volume if the updating criterion Eq. (14) is satisfied, and replace k by $k+1$ and return to step 2.

Step 9: Terminate the process if the stopping criterion Eq. (16) or Eq. (17) is satisfied. Otherwise, replace k by $k+1$ and return to step 2.

Results and Discussions

In this section, two cases with the triangular and rectangular profile of the longitudinal fin are deal with to demonstrate the proposed method. The two-dimensional model will be considered in two cases. Additionally, the optimal results by the proposed method are discussed and compared with the theory results by Kruas [1]. The longitudinal fin with the height of fin of $H = 0.2[\text{m}]$ and the thermal conductivity of $k = 58.3[\text{W/mK}]$ is considered in two cases. It is assumed that our purpose is to find the minimum volume of the fin so that the fin can dissipate a given heat flow of $Q = 20[W]$ with the base temperature of $T_b = 400[K]$ in the surrounding ambient with the temperature $T_a = 300[K]$. The convective heat transfer coefficient is considered to be constants and obtained from Eq. (18) by Dobaru [13] as following:

$$h = \frac{8k\,\text{Pr}^{1/2}}{3H\left[336\left(\text{Pr}+\frac{9}{5}\right)\right]}\left(\frac{g\beta\left[T(x)-T_\infty\right]H^3}{v^2}\right)^{1/4} \tag{18}$$

where, all the fluid properties are computed at a mean temperature, $T_m = (T_b + T_a)/2$. With the given thermal parameters above, the mean convective heat transfer coefficient is $\bar{h} = 5.2564\ [\text{W/mK}^2]$.

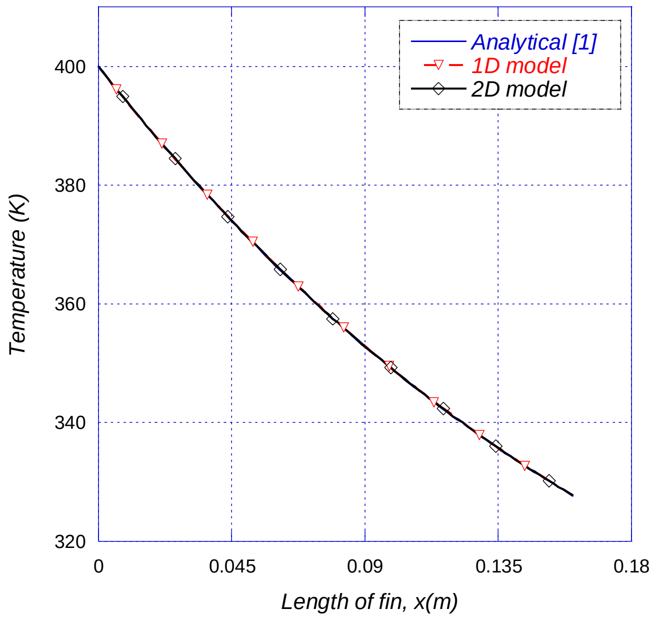

Figure 3. The temperature distribution along the length of the optimal fin for triangle profile.

Table 1. The geometrical parameters of the optimal fin by MRN method and theoretical results for the triangle profile

Dimension of optimum fin	Theoretical Results	The proposed Method
The length, *x[m]*	$x = 1.6022e-1$	$x = 1.6020e-1$
The semi-width, *y[m]*	$y = 1.3498e-3$	$y = 1.3503e-3$
The min volume, *V[m^3]*	$V = 4.3255e-5$	$V = 4.3265e-5$

Table 2. The relative difference of fin geometrical parameters between MRN method and theoretical results for the triangle profile

Dimension of optimum fin	Error (%)
The length, %*x*	0.01%
The semi-width, %*y*	0.04%
The min volume, %*V*	0.023%

In Case 1, a fin design problem with triangular profile of the longitudinal fin is considered. For optimization procedure by MNR, the value of updating and stopping criteria used were $\delta = 10^{-4}$ and $\varepsilon = 10^{-4}$ respectively. The initial volume is $V_{xpcd}^{0} = 4e-5(m^3)$. The initial dimensions of the triangular profile fin are $x_0 = 0.3[m]$ and $y_0 = 0.001[m]$. In this case, the optimal results obtained by the proposed method and the theory optimal results are shown in Table 1. The relative difference of the geometrical parameters of the optimal fin between the theoretical values and MNR's results are also shown in Table 2. Furthermore, the temperature distribution along the length of the optimal triangular fin is presented in Figure 3.

As shown, with specified thermal properties and given boundary conditions above, the minimum volume of the optimal triangular fin is about $V = 4.325e-5$. The optimal length of triangular fin is about $x = 0.16[m]$ and the optimal semi-width triangular fin is about $y = 1.35e-3[m]$. Table 2 show that the relative error of the geometrical parameters of the optimum fin between MRN method and the theoretical is small. The relative error for the minimum volume is 0.23% and that for the length and semi-width are 0.01% and 0.04% respectively. This mean that the results obtained by the proposed method satisfied the given condition and are in high agreement with the theoretical values.

Table 3. The geometrical parameters of the optimal fin by MRN method and theoretical results for the rectangle profile

Dimension of optimum fin	Theoretical result	The proposed Method	
		2D model (no tip convection)	2D model (tip convection)
The length, *x[m]*	$x = 1.5178e-1$	$x = 1.5179e-1$	$x = 1.5022e-1$
The semi-width, *y[m]*	$y = 1.0313e-3$	$y = 1.0313e-3$	$y = 1.0350e-3$
The min volume, *V[m^3]*	$V = 6.2613e-5$	$V = 6.2614e-5$	$V = 6.2188e-5$

Table 4. The relative difference of fin geometrical parameters between MRN method and theoretical results for the rectangle profile

Dimension of optimum fin	The proposed Method	
	2D model (without convective tip)	2D model (convective tip)
The length, %x	0.02%	1%
The semi-width, %y	0%	0.3%
The min volume, %V	0.0016%	0.68%

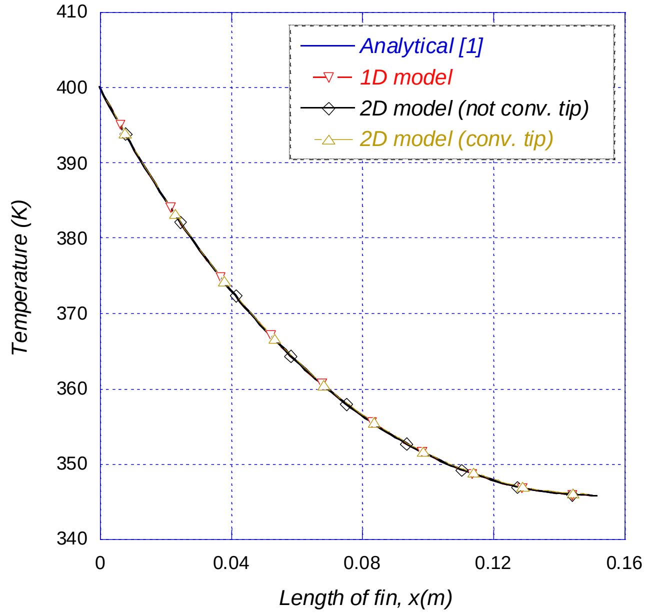

Figure 4. The temperature distribution along the length of the optimal fin for the rectangle profile.

In Case 2, the fin design problem with the rectangular profile is investigated. In this case, the initial dimensions of the rectangle fin are $x_0 = 0.2[m]$ and $y_0 = 0.001[m]$. Two cases of convective tip and insulated tip are considered in this case. Table 3 showed the optimal results achieved by the proposed method and the theoretical formulation. Table 4 illustrated the relative deviation of the geometrical parameters of the optimal rectangular fin between MNR's method and theoretical formulation. In addition, the temperature distribution along the length of the optimal rectangular fin is drawn in Figure 4.

The obtained results showed that there is good approximation between the optimal result by MNR's method and theoretical method for the case of insulated tip. Particularly, the minimum volume the case of insulated tip is about $V = 6.26e-5[m^3]$ for both of the theory method and the proposed method. The length and semi-width of the optimal rectangular fin are respectively about $x = 1.52e-1[m]$ and about $y = 1.03e-3[m]$ with the very small relative deviation between the methods (as Table 4). For the case of the convective tip, the minimum volume of optimal fin is about $V = 6.22e-5[m^3]$. As shown in Table 4, the value of the optimal rectangular fin volume with the convective tip is 0.68% less than that with the insulated tip. This is due to the face that the consideration of convective tip leads the increase of heat dissipation comparing with the assumption of the insulated tip. Thus, the volume of the optimal rectangle fin with the convective tip is less than that with the insulated tip.

With the obtained results from two cases, it can be said that the proposed method is potential and feasible in finding the minimum volume of the optimum fin with rectangular and triangular profile. Furthermore, the proposed method does not depend on the type of the direct problem (linear or non-linear direct problem). In the other words, the proposed method can be utilized in finding the minimum volume for any fin design problems with rectangular and triangular profile.

Conclusions

In this work, the minimum volume of the longitudinal fin with the rectangular and triangular profile for the given heat flow and the expected temperature at the base by using the modified Newton Raphson method was presented. A mechanism called as "volume updating" was added in the proposed algorithm to obtain the minimum volume of the optimum fin. Two cases with the rectangle and triangle profile were performed to validate the proposed method. The obtained results by MNR's method have been compared with the results of Kraus [1]. The results showed that the values of the volume of the optimal fin are in good agreement with that of Kraus [1] in all two cases. In the other words, it can be declared that the proposed method is an efficient and accurate method to find the minimum volume of the optimal fin with triangle and rectangle profile for the given heat flow and the expected temperature at the base. Furthermore, the proposed method does not depend upon the type of the direct problem. Thus, this method can be applied for any linear or non-linear fin design problem.

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