Complex normal form method for nonlinear free vibration of a cantilever nan-obeam with surface effects

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Abstract

Nano-beams and nanowires are widely used as building blocks in the rapid development of Nano/Micro-electro-mechanical system (N/MEMS), micro-sensors, energy harvesting and storage devices, etc., and their vibration behaviors have aroused great concerns in both pure science and engineering applications. In this study, we investigate the nonlinear free vibration of a nano-beam considering its surface effects, including the surface elasticity and the residual surface stress. Firstly, a mechanics model on the transverse vibration of a cantilever nano-beam is developed according to Hamilton's principle. In use of the Galerkin and complex normal form methods, the approximate analytical solution of the nonlinear equation is obtained, which has been confirmed by the numerical simulation. The present work can provide theoretical basis for the precise design of nanowires or nanofibers in atomic force microscopy, generators and nano-sensors in electronic devices.

Keywords: Surface elasticity， Residual surface stress， Complex normal form method， Quasi-periodic motion， Chaos

1 Introduction

Nano-beams or nanowires, due to their perfect advantages of greatly magnified sensitivity, increased reliability and reduced sizes, have been highly advanced in Nano/Micro-electro-mechanical systems (N/MEMS), biotechnology, sensors, actuators, resonators and atomic force microscopy [1-3]. Owing to the extremely high surface to volume ratio, surface effects of nanowires have become more important factors than the volumetric forces, which are crucial to their mechanical performance. Based on a number of results from experiments and atomic simulations, the residual surface stress and surface elasticity have proved to be the key origins of the size-dependent properties for most nanomaterials or nanostructures [4].

It has already been known that surface effects can significantly affect the static properties of nanowires, such as the internal force diagram, modulus, deflection and buckling of nano-beams [5–8], and these phenomena have attracted growing interest of many scholars. However, the more interesting issue for a nano-beam is its dynamic response, because it is an essential techinique to measure the dynamic parameters in vibration, in order to characterize the bending stiffness [1]. As a consequence, a great deal of work has been performed to investigate the natural frequency and amplitude of a nano-beam in linear vibration [9–13]. For example, Wang and Feng [9] deduced the natural frequency of a simply supported nano-beam, in consideration of both the surface elasticity and residual surface stress, which indicates that the frequency may be enhanced with a positive residual surface stress and reduced by a negative one. Based on the same model, He and Lilley [10] studied the influence of surface effects on the first-mode natural frequency for a nano-beam with different boundary conditions. Their theoretical solutions show that the positive surface stress can alter the natural frequency for a cantilever, a simply supported beam and a fixed-fixed nano-beam. From the different viewpoint, Ansari and Hosseini *et al*. [11] explored the impact of surface effects on the natural frequency of a nano-beam in use of the compact finite difference method. They claimed that the surface effects on the natural frequency are dependent on the aspect ratio and thickness of the beam. It should be stressed that for a nano-beam with small aspect ratio, the Timoshenko beam model is normally utilized to analyze its transverse vibration, which is more accurate than the Euler-Bernoulli beam model mentioned previously [12, 13].

Moreover, it is necessary to consider the nonlinear vibration of nanowires in many applications [14, 15]. For example, it is desirable to reduce the size of N/MEMS and achieve high-output energy, but this requires the nano-beam or nanowire in N/MEMS come into operation near the nonlinear working regime [14]. For instance, Chen and Hu *et al*. [16] developed a periodicity-ratio approach to calculate the response of a piezoelectric laminated micro-beam system actuated by AC and DC voltages, and the periodic and chaotic region diagrams were plotted. Miandoab and Yousefi-Koma *et al*. [17] studied the chaotic behaviors of a nano-resonator in MEMS/NEMS subjected to electrostatic forces, and they found that the system undergoes homoclinic and heteroclinic bifurcations in the appearance of chaos. Yet in the real situations, besides such normal nonlinear factors as curvature, geometry nonlinearities and the coupling of multi-fields [18–21], surface effects play an important role in the nonlinear vibration of nano-beams. One example is that, taking the von-Karman geometric nonlinear strain into account, Gheshlaghi and Hasheminejad [22] studied the influence of residual surface stress on the free nonlinear vibration of a nano-beam and got the exact expressions of the natural frequency and vibration amplitude. In addition, Moeenfard and Mojahedi *et al*. [23] used the Homotopy Perturbation Method to analyze the nonlinear free vibration of a clamped-clamped or a clamped–free nano-beam, and the effects of axial loads, rotary inertia, shear deformation and slenderness ratio on the natural frequency have been discussed.

It should be stressed that, to fully probe the dynamics of a nano-beam, it is imperative to consider its semi-analytical solution, and more importantly, surface effects are necessary to be analyzed in this model. To the best of our knowledge, there is hitherto a lack of systematic exploration on this issue, for there is a very strong coupling between the nonlinear factors and surface effects. Therefore, we concentrate on the nonlinear free vibration of a nano-beam, towards extending our understandings on the solution, which is helpful to better design N/MEMS, micro-sensors, energy harvesting and storage devices.

The present paper is organized as follows. In Section 2, the dynamics equation of the free vibration of a cantilever beam including surface effects is derived based on Hamilton Principle. The Galerkin method is then adopted to discrete the partial differential equation into ordinary differential equations in Section 3. Next, in Section 4, we analyze the solution and its stability on the nonlinear vibration by using the complex normal form method (CNFM), and the numerical simulation is followed. Finally, conclusion are given in detail.

2 Kinematics

2.1 Surface effects model

Generally speaking, “surface effects” of nano-materials are mainly attributed to the surface energy or surface stresses on the solid surface [24, 25]. The body and surface layer of a nanowire can be abstracted as a composite beam with a core-shell structure, which includes a solid core with a Young's modulus and a surface layer with a surface modulus [9], as schematized in Fig. 1. The thickness of the surface layer is normally negligible.

In light of the generalized Young–Laplace equation [9], the residual surface stress in the surface layer of the nano-beam can induce a jump of the normal stress across the interface between the bulk and the surface, and this leads to a transversely distributed pressure $q_s(s)$ along the axial direction of the beam, namely

$$q_s(s) = H\kappa, \tag{1}$$

where H is a constant parameter correlated with the residual surface stress and the cross-sectional shape. The parameter H for a nano-beam with a circular cross section is normally expressed as $H = 2\tau_0 D$ [9], where D is the diameter of the cross section.

For the nano-beam with a circular cross section, the effective bending stiffness $(EI)^*$ can be further modified according to the composite beam model, which is expressed as [6, 9, 10]

$$(EI)^* = \frac{\pi}{64} ED^4 + \frac{\pi}{8} E^s D^3, \tag{2}$$

and the effective tensile stiffness $(EA)^*$ is given by

$$(EA)^* = \frac{\pi}{4} ED^2 + E^s \pi D, \tag{3}$$

where E is the Young's modulus of the bulk material, E^s the surface elastic modulus, I the moment of inertia on the cross section, and A is the area of the cross section.

2.2 Vibration equation

We consider a cantilever nano-beam with surface effects, as shown in Fig. 1. The length of the beam is L, and the mass per unit length is m. Refer to a Cartesian coordinate system (O-xy). The model is assumed to be an Euler-Bernoulli beam, whose axis is initially along the x direction and then it can oscillate in the (x, y) plane [26]. As schematized in Fig. 1, the location of an arbitrary material point B in the beam axis transfers to the position of point B_1 after deformation. Let ϕ be the slope angle between the tangential line of the beam axis and the horizontal line at any point in the axis. We also introduce the curvilinear coordinate, i.e. the arc length s along the axis of the beam, starting from the origin.

To analyze the nonlinear vibration of the beam, the nonlinear effects on the deformation must be incorporated, so the infinitesimal deformation model can not be adopted here. In fact, at any point in the beam axis, there are the following geometric relations:

$$\cos\phi = x', \ \sin\phi = y'. \tag{4}$$

Taking derivative with respect to the arc length s on both sides of the second equation in Eq. (4), one can get the expression of the planar curvature

$$\kappa = \phi' = \frac{1}{\cos\phi} y'' = \frac{y''}{\sqrt{1-y'^2}}, \tag{5}$$

where $(\ )' = \frac{\mathrm{d}(\ )}{\mathrm{d}s}$ and $(\ )'' = \frac{\mathrm{d}^2(\ )}{\mathrm{d}s^2}$.

In use of the Hamilton's principle, one has

$$\int_0^T \delta L(y,T)\mathrm{d}T + \int_0^T \delta W(y,T)\mathrm{d}T = 0, \tag{6}$$

where time is denoted by T and the Lagrangian function is L=U–K.

The strain energy, kinetic energy and external work of the beam can be respectively given by

$$U=\frac{(EI)^*}{2}\int_0^L \kappa^2\mathrm{d}s=\frac{(EI)^*}{2}\int_0^L \frac{y''^2}{1-y'^2}\mathrm{d}s\,,\ K=\frac{m}{2}\int_0^L\left(\frac{\mathrm{d}y}{\mathrm{d}T}\right)^2\mathrm{d}s\,,\ W=\int_0^L q_s y\mathrm{d}s\,. \qquad (7)$$

By virtue of the variational principle, the governing equation of the free vibration for the nano-beam can be deduced as

$$m\ddot{y}+(EI)^*\left(y''''+y''''y'^2+4y'y''y'''+y''^3\right)-H\left(y''+\frac{1}{2}y'^2y''\right)=0\,. \qquad (8)$$

The initial conditions and fixed boundary conditions of the beam are $y(0,T)=0$, $y'(0,T)=0$, $y''(L,T)=0$, $y'''(L,T)=0$.

Introducing the following non-dimensional quantities $w=\frac{y}{L}$, $\xi=\frac{s}{L}$, $\omega^*=\frac{1}{L^2}\sqrt{\frac{(EI)^*}{m}}$, $t=\omega^*T$, $\beta=\frac{HL^2}{(EI)^*}$, the governing equation can be recast as

$$\ddot{w}+w^{(4)}-\beta\left(w''+\frac{1}{2}w'^2w''\right)+\left(w''''w'^2+4w'w''w'''+w''^3\right)=0\,, \qquad (9)$$

where $\dot{w}=\frac{\mathrm{d}w}{\mathrm{d}t}$, $\ddot{w}=\frac{\mathrm{d}^2w}{\mathrm{d}t^2}$, $w'=\frac{\mathrm{d}w}{\mathrm{d}\xi}$, $w''=\frac{\mathrm{d}^2w}{\mathrm{d}\xi^2}$, $w'''=\frac{\mathrm{d}^3w}{\mathrm{d}\xi^3}$, and $w^{(4)}=\frac{\mathrm{d}^4w}{\mathrm{d}\xi^4}$.

3 Galerkin Method

It is noticed that Eq. (9) is a high order and nonlinear partial differential equation (PDE), and it is nearly impossible to find the close-formed solution at hand. Herein, we use the Garlerkin discretization method to transform the PDE to the ordinary differential equations (ODEs).

We select the two-mode approximation on the solution, which is of adequately accuracy to analyze the nonlinear vibration. Assumes $w(\xi,t)$ can be approximated by the two-mode solution as below:

$$w(\xi,t)=q_1(t)\phi_1(\xi)+q_2(t)\phi_2(\xi)\,, \qquad (10)$$

where $q_1(t)$ and $q_2(t)$ represent the amplitudes of the first and second principal modes, respectively. The first and the second mode shape functions $\phi_1(\xi)$ and $\phi_2(\xi)$ can be described as:

$$\phi_i(\xi)=\cosh r_i\xi-\cos r_i\xi+\frac{\sin r_i-\sinh r_i}{\cos r_i+\cosh r_i}\left(\sinh r_i\xi-\sin r_i\xi\right)\ (i=1,\,2), \qquad (11)$$

where r_i is governed by the frequency equation on a cantilever beam: $\cos(r_i)\cos\mathrm{h}(r_i)=-1$. We first substitute Eqs. (10) and (11) into (9), and the obtained equation is multiplied by $\phi_1(\xi)$ or $\phi_2(\xi)$, then the integration of the product from 0 to 1 yields a second-order differential equation, where the orthogonal property of trigonometric functions is used. As a result, the ODEs on $(q_1,q_2)^{\mathrm{T}}$ are presented as:

$$\begin{cases}\ddot{q}_1+a_{11}q_1+a_{12}q_2+\left(a_{13}q_1^3+a_{14}q_1^2q_2+a_{15}q_1q_2^2+a_{16}q_2^3\right)=0\\ \ddot{q}_2+a_{21}q_1+a_{22}q_2+\left(a_{23}q_1^3+a_{24}q_1^2q_2+a_{25}q_1q_2^2+a_{26}q_2^3\right)=0\end{cases}, \qquad (12)$$

where the values of the parameters a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26} are shown in Appendix A.

4 Solution analysis: CNFM

Next, we use the CNFM to solve the nonlinear ODEs in Eq. (12), where the two quantities q_1 and q_2 are coupled together [27]. Firstly, we assume its solutions can be formulated as

$$\begin{cases} q_1 = \zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2 \\ q_2 = \Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right) \end{cases}, \tag{13}$$

where $\bar{\zeta}_1$, $\bar{\zeta}_2$ are the complex conjugates of ζ_1 and ζ_2, respectively, and their normal expressions are

$$\zeta_i = A_i e^{j\omega_i t}\ ,\ \bar{\zeta}_i = \bar{A}_i e^{-j\omega_i t}\ \ (i\text{=1,2}), \tag{14}$$

where ω_1 and ω_2 are two frequencies, and $j^2 = -1$.

The first step is to insert Eq. (13) into the linear part of Eq. (12), and then we can derive the frequency equation on ω_i

$$\omega^4 - \left(a_{11} + a_{22}\right)\omega^2 + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{15}$$

Moreover, the parameters Δ_1 and Δ_2 are given as

$$\Delta_i = -\frac{a_{11} - \omega_i^2}{a_{12}} = -\frac{a_{21}}{a_{22} - \omega_i^2}\ \ (i\text{=1,2}), \tag{16}$$

which are both real numbers.

In combination with Eq. (13) and its derivatives with respect to t, it can reach an equation group including four equations. Consequently, the complex variable equations on the parameters ζ_1 and ζ_2 are listed as

$$\dot{\zeta}_1 = j\omega_1\zeta_1 - j\left\{\begin{aligned} & b_{11}\left(\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right)^3 \\ & + b_{12}\left(\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right)^2\left(\Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right)\right) \\ & + b_{23}\left(\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right)\left(\Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right)\right)^2 \\ & + b_{14}\left(\Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right)\right)^3 \end{aligned}\right\}, \tag{17}$$

$$\dot{\zeta}_2 = j\omega_2\zeta_2 - j\left\{\begin{aligned} & b_{21}\left(\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right)^3 \\ & + b_{22}\left(\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right)^2\left(\Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right)\right) \\ & + b_{23}\left(\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right)\left(\Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right)\right)^2 \\ & + b_{24}\left(\Delta_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Delta_2\left(\zeta_2 + \bar{\zeta}_2\right)\right)^3 \end{aligned}\right\}, \tag{18}$$

where the expressions of b_{11}, b_{12}, b_{13}, b_{14}, b_{21}, b_{22}, b_{23}, b_{24} are all shown in Appendix B.

In order to simplify the above equation, we introduce the following near-identity coordinate transformations

$$\begin{cases} \zeta_1 = \eta_1 + h_1\left(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2\right) \\ \zeta_2 = \eta_2 + h_2\left(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2\right) \end{cases}, \tag{19}$$

where the functions h_1 and h_2 are expressed in Appendix C.
Inserting Eq. (19) into Eqs. (17) and (18), and if the values of $\Lambda_{1,i}$, $\Lambda_{2,i}$ (i=1, …, 20) in Eq. (19) are properly chosen (shown in Appendix C), one can get the simplest normal forms of Eqs. (17) and (18):

$$\begin{cases} \dot{\eta}_1 = j\omega_1\eta_1 - \varepsilon j\left(c_{11}\eta_1\eta_2\bar{\eta}_2 + c_{12}\eta_1^2\bar{\eta}_1\right) \\ \dot{\eta}_2 = j\omega_2\eta_2 - \varepsilon j\left(c_{21}\eta_1\bar{\eta}_1\eta_2 + c_{22}\eta_2^2\bar{\eta}_2\right) \end{cases}, \tag{20}$$

where ε is small perturbation parameter, and the expressions of the symbols c_{11}, c_{12}, c_{21}, c_{22} are all given in Appendix D.

If the quasi-periodic solutions of the equation exist, the expression of η_1 and η_2 can be written in the polar form

$$\begin{cases} \eta_1 = \dfrac{1}{2}a_1 \exp j\left(\omega_1 t + \theta_1\right) \\ \eta_2 = \dfrac{1}{2}a_2 \exp j\left(\omega_2 t + \theta_2\right) \end{cases}, \tag{21}$$

where a_1, a_2 are amplitudes and θ_1 and θ_2 are phase angles, which are all real numbers.

Substituting Eq. (21) into Eq. (20), and separating the real and imaginary parts yields

$$\begin{cases} \dot{a}_1 = 0 \\ \dot{a}_2 = 0 \\ a_1\dot{\theta}_1 = -\dfrac{1}{4}a_1 a_2^2 c_{11} - \dfrac{1}{4}a_1^3 c_{12} \\ a_2\dot{\theta}_2 = -\dfrac{1}{4}a_1^2 a_2 c_{21} - \dfrac{1}{4}a_2^3 c_{22} \end{cases}, \tag{22}$$

The above equations tells us that, a_1 and a_2 must be constants, and θ_1 and θ_2 can be solved when the a_1 and a_2 are given.

Up to now, the solutions on q_1 and q_2 can be acquired, and their expressions are shown in Appendix E. Therefore, the embryo conclusion is that, since a_1 and a_2 can be taken as arbitrary values, the considered system must have multiple quasi-periodic trajectories; and with different initial conditions, the system will converge to different quasi-periodic trajectories.

5 Validation of CNFM

To verify the validity of the CNFM, the numerical simulation based on the Runge-Kutta method is performed, where the time span is selected from 0 to 40 seconds. In the simulation process, the physical parameters of the nano-beam are chosen as [5, 8]: D=50 nm, L=500 nm, the mass per meter m=3.7876$\times 10^{-11}$ kg/m, the bending stiffness $(EI)^*$ =2.33$\times 10^{-20}$ $\mathrm{Pa\cdot m^4}$ and tensile stiffness $(EA)^*$=1.49$\times 10^{-4}$ $\mathrm{Pa\cdot m^4}$. The residual surface stress τ_0, which can be either positive or negative depending on the crystallographic structure for different nanomaterials, is selected in the regime from –2 to 2 N/m [8]. The simulation program is realized in MATLAB using the 4$^{\text{th}}$-order Runge-Kutta method, where the time step is set as 0.02 seconds.

We only study the trivial singular point near the center $\left(a_2,\theta_1,a_2,\theta_2\right)$=(0, 0, 0, 0), where multiple stable quasi-periodical solutions exist. For the CNFM, the parameters are selected as: a_1=0.005, a_2=0.0005. Correspondingly, the initial parameters in the numerical simulation are selected as $(q_1,\dot{q}_1,q_2,\dot{q}_2)\big|_{t=0}$ =(0.0055, 0, 0.00129, 0) to ensure they have the same initial values as those of the CNFM. The time history diagrams on q_1 and q_2 are shown in Fig. 2(a) and (b), respectively. From the figure it is clearly seen that the two curves from the Runge-Kutta method and CNFM nearly overlap with each other. This manifests that the CNFM is efficient to analyze the vibration of this system with small magnitude $|q_1|<0.01$ and $|q_2|<0.01$, as this method is applicable in weak nonlinear systems.

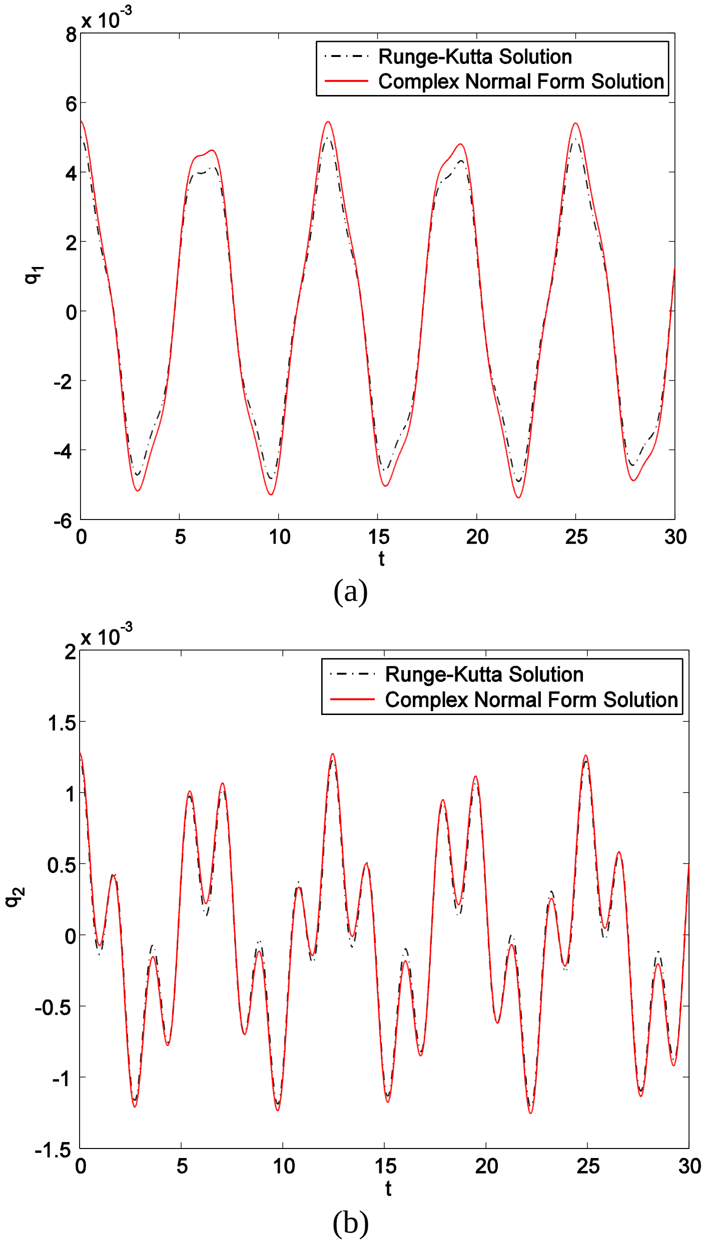

Fig. 2 Comparison between the complex normal form solution and the Runge-Kutta solution, (a) q_1, (b) q_2.

6 Conclusion

In conclusion, the nonlinear free vibration of a cantilever nano-beam has been systematically investigated, and the surface effects are considered. The CNFM and numerical simulation are applied to obtain the solution of the system. The numerical simulation demonstrates that the complex normal form can be accurate enough to analysis the vibration with small magnitude The study provides insight into the mechanism of the nonlinear dynamics of nanowires, and given theoretical basement for design of the element structures in N/MEMS, sensors, actuators, and resonators, etc.

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Appendix A

In Section 3, the parameters of Eq. (12) are shown as follows:

$a_{11} = 0.0082 - 0.8588\beta_1$,

$a_{12} = -0.0312 + 11.7440\beta_1$,

$a_{13} = 40.4229 - 2.1570\beta_1$,

$a_{14} = -306.8797 + 41.2253\beta_1$,

$a_{15} = 2484.0513 - 147.2018\beta_1$ $a_{15} = 2484.0513 - 147.2018\beta_1$,

$a_{16} = -2178.2983 + 192.7033\beta_1$,

$a_{22} = 0.1131 + 13.2933\beta_1$,

$a_{21} = -0.02992 - 1.8738\beta_1$,

$a_{23} = -102.2965 - 3.4639\beta_1$,

$a_{24} = 2484.1298 + 32.0625\beta_1$,

$a_{25} = -6530.1573 - 44.4370\beta_1$,

$a_{26} = 13416.8369 + 53.2163\beta_1$.

Appendix B

In Section 4, the parameters of Eqs. (17) and (18) are shown as follows:

$$b_{11} = \frac{a_{23} - \Delta_2 a_{13}}{2\omega_1(\Delta_2 - \Delta_1)}, b_{12} = \frac{a_{24} - \Delta_2 a_{14}}{2\omega_1(\Delta_2 - \Delta_1)}, b_{13} = \frac{a_{25} - \Delta_2 a_{15}}{2\omega_1(\Delta_2 - \Delta_1)}, b_{14} = \frac{a_{26} - \Delta_2 a_{16}}{2\omega_1(\Delta_2 - \Delta_1)},$$

$$b_{21} = \frac{\Delta_1 a_{13} - a_{23}}{2\omega_2(\Delta_2 - \Delta_1)}, b_{22} = \frac{\Delta_1 a_{14} - a_{24}}{2\omega_2(\Delta_2 - \Delta_1)}, b_{23} = \frac{\Delta_1 a_{15} - a_{25}}{2\omega_2(\Delta_2 - \Delta_1)}, b_{24} = \frac{\Delta_1 a_{16} - a_{26}}{2\omega_2(\Delta_2 - \Delta_1)}.$$

Appendix C

$$\begin{aligned} h_2(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2) &= \Lambda_{2,1}\eta_1^3 + \Lambda_{2,2}\bar{\eta}_1^3 + \Lambda_{2,3}\eta_2^3 + \Lambda_{2,4}\bar{\eta}_2^3 + \Lambda_{2,5}\eta_1^2\bar{\eta}_1 \\ &+ \Lambda_{2,6}\eta_1^2\eta_2 + \Lambda_{2,7}\eta_1^2\bar{\eta}_2 + \Lambda_{2,8}\eta_1\bar{\eta}_1^2 + \Lambda_{2,9}\eta_1\eta_2^2 + \Lambda_{2,10}\eta_1\bar{\eta}_2^2 + \Lambda_{2,11}\bar{\eta}_1^2\eta_2 \\ &+ \Lambda_{2,12}\bar{\eta}_1^2\bar{\eta}_2 + \Lambda_{2,13}\bar{\eta}_1\eta_2^2 + \Lambda_{2,14}\bar{\eta}_1\bar{\eta}_2^2 + \Lambda_{2,15}\eta_2^2\bar{\eta}_2 + \Lambda_{2,16}\eta_2\bar{\eta}_2^2 \\ &+ \Lambda_{2,17}\bar{\eta}_1\eta_2\bar{\eta}_2 + \Lambda_{2,18}\eta_1\eta_2\bar{\eta}_2 + \Lambda_{2,19}\eta_1\bar{\eta}_1\bar{\eta}_2 + \Lambda_{2,20}\eta_1\bar{\eta}_1\eta_2. \end{aligned},$$

$$\begin{aligned} h_1(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2) &= \Lambda_{1,1}\eta_1^3 + \Lambda_{1,2}\bar{\eta}_1^3 + \Lambda_{1,3}\eta_2^3 + \Lambda_{1,4}\bar{\eta}_2^3 + \Lambda_{1,5}\eta_1^2\bar{\eta}_1 \\ &+ \Lambda_{1,6}\eta_1^2\eta_2 + \Lambda_{1,7}\eta_1^2\bar{\eta}_2 + \Lambda_{1,8}\eta_1\bar{\eta}_1^2 + \Lambda_{1,9}\eta_1\eta_2^2 + \Lambda_{1,10}\eta_1\bar{\eta}_2^2 + \Lambda_{1,11}\bar{\eta}_1^2\eta_2 \\ &+ \Lambda_{1,12}\bar{\eta}_1^2\bar{\eta}_2 + \Lambda_{1,13}\bar{\eta}_1\eta_2^2 + \Lambda_{1,14}\bar{\eta}_1\bar{\eta}_2^2 + \Lambda_{1,15}\eta_2^2\bar{\eta}_2 + \Lambda_{1,16}\eta_2\bar{\eta}_2^2 \\ &+ \Lambda_{1,17}\bar{\eta}_1\eta_2\bar{\eta}_2 + \Lambda_{1,18}\eta_1\eta_2\bar{\eta}_2 + \Lambda_{1,19}\eta_1\bar{\eta}_1\bar{\eta}_2 + \Lambda_{1,20}\eta_1\bar{\eta}_1\eta_2; \end{aligned},$$

$$\Lambda_{1,1} = -\frac{b_{11} + b_{12}\Delta_1 + b_{13}\Delta_1^2 + b_{14}\Delta_1^3}{2\omega_1}, \quad \Lambda_{1,2} = -\frac{\Lambda_{1,1}}{2}, \quad \Lambda_{1,3} = -\frac{b_{11} + b_{12}\Delta_2 + b_{13}\Delta_2^2 + b_{14}\Delta_2^3}{\omega_1 - 3\omega_2},$$

$$\Lambda_{1,4}=\frac{b_{11}+b_{12}\Delta_2+b_{13}\Delta_2^2+b_{14}\Delta_2^3}{\omega_1+3\omega_2},\ \Lambda_{1,5}=-\Lambda_{1,8},$$

$$\Lambda_{1,8}=\frac{3b_{11}+3\Delta_1 b_{12}+3\Delta_1^2 b_{13}+3\Delta_1^3 b_{14}}{2\omega_1},\ \Lambda_{1,6}=-\frac{3b_{11}+b_{12}\left(2\Delta_1+\Delta_2\right)+b_{13}\left(\Delta_1^2+2\Delta_1\Delta_2\right)+3\Delta_1^2\Delta_2 b_{14}}{\omega_1+\omega_2},$$

$$\Lambda_{1,7}=\frac{3b_{11}+b_{12}\left(2\Delta_1+\Delta_2\right)+b_{13}\left(\Delta_1^2+2\Delta_1\Delta_2\right)+3\Delta_1^2\Delta_2 b_{14}}{\omega_2-\omega_1},$$

$$\Lambda_{1,9}=-\frac{3b_{11}+b_{12}\left(\Delta_1+2\Delta_2\right)+b_{13}\left(\Delta_2^2+2\Delta_1\Delta_2\right)+3\Delta_1\Delta_2^2 b_{14}}{2\omega_2},$$

$$\Lambda_{1,10}=-\Lambda_{1,9},$$

$$\Lambda_{1,11}=\frac{3b_{11}+b_{12}\left(2\Delta_1+\Delta_2\right)+b_{13}\left(\Delta_1^2+2\Delta_1\Delta_2\right)+3\Delta_1^2\Delta_2 b_{14}}{3\omega_1-\omega_2},$$

$$\Lambda_{1,12}=\frac{3b_{11}+b_{12}\left(2\Delta_1+\Delta_2\right)+b_{13}\left(\Delta_1^2+2\Delta_1\Delta_2\right)+3\Delta_1^2\Delta_2 b_{14}}{3\omega_1+\omega_2},$$

$$\Lambda_{1,13}=-\frac{3b_{11}+b_{12}\left(\Delta_1+2\Delta_2\right)+b_{13}\left(\Delta_2^2+2\Delta_1\Delta_2\right)+3\Delta_1\Delta_2^2 b_{14}}{2\omega_1-2\omega_2},$$

$$\Lambda_{1,14}=-\frac{3b_{11}+b_{12}\left(\Delta_1+2\Delta_2\right)+b_{13}\left(\Delta_2^2+2\Delta_1\Delta_2\right)+3\Delta_1\Delta_2^2 b_{14}}{2\omega_1+2\omega_2},$$

$$\Lambda_{1,15}=\frac{3b_{11}+3\Delta_2 b_{12}+3\Delta_2^2 b_{13}+3\Delta_2^3 b_{14}}{\omega_1-\omega_2},$$

$$\Lambda_{1,16}=\frac{3b_{11}+3\Delta_2 b_{12}+3\Delta_2^2 b_{13}+3\Delta_2^3 b_{14}}{\omega_1+\omega_2},\ \Lambda_{1,17}=\frac{3b_{11}+b_{12}\left(\Delta_1+2\Delta_2\right)+b_{13}\left(\Delta_2^2+2\Delta_1\Delta_2\right)+3\Delta_1\Delta_2^2 b_{14}}{\omega_1},$$

$$\Lambda_{1,18}=-\Lambda_{1,17},$$

$$\Lambda_{1,19}=\frac{6b_{11}+b_{12}\left(4\Delta_1+2\Delta_2\right)+b_{13}\left(2\Delta_1^2+4\Delta_1\Delta_2\right)+6\Delta_1^2\Delta_2 b_{14}}{\omega_1+\omega_2},$$

$$\Lambda_{1,20}=\frac{6b_{11}+b_{12}\left(4\Delta_1+2\Delta_2\right)+b_{13}\left(2\Delta_1^2+4\Delta_1\Delta_2\right)+6\Delta_1^2\Delta_2 b_{14}}{\omega_1-\omega_2},$$

$$\Lambda_{2,1}=\frac{b_{21}+b_{22}\Delta_1+b_{23}\Delta_1^2+b_{24}\Delta_1^3}{\omega_2-3\omega_1},\ \Lambda_{2,2}=\frac{b_{21}+b_{22}\Delta_1+b_{23}\Delta_1^2+b_{24}\Delta_1^3}{\omega_2+3\omega_1},$$

$$\Lambda_{2,3}=-\frac{b_{21}+b_{22}\Delta_2+b_{23}\Delta_2^2+b_{24}\Delta_2^3}{2\omega_2},\ \Lambda_{2,4}=\frac{b_{21}+b_{22}\Delta_2+b_{23}\Delta_2^2+b_{24}\Delta_2^3}{4\omega_2},$$

$$\Lambda_{2,5}=\frac{3b_{21}+3\Delta_1 b_{22}+3\Delta_1^2 b_{23}+3\Delta_1^3 b_{24}}{\omega_2-\omega_1},\ \Lambda_{2,8}=\frac{3b_{21}+3\Delta_1 b_{22}+3\Delta_1^2 b_{23}+3\Delta_1^3 b_{24}}{\omega_1+\omega_2},$$

$$\Lambda_{2,6}=-\frac{3b_{21}+b_{22}\left(2\Delta_1+\Delta_2\right)+b_{23}\left(\Delta_1^2+2\Delta_1\Delta_2\right)+3\Delta_1^2\Delta_2 b_{24}}{2\omega_1},$$

$$\Lambda_{2,7}=\frac{3b_{21}+b_{22}\left(2\Delta_1+\Delta_2\right)+b_{23}\left(\Delta_1^2+2\Delta_1\Delta_2\right)+3\Delta_1^2\Delta_2 b_{24}}{2\omega_2-\omega_1},$$

$$\Lambda_{2,9} = -\frac{3b_{21} + b_{22}\left(\Delta_1 + 2\Delta_2\right) + b_{23}\left(\Delta_2^2 + 2\Delta_1\Delta_2\right) + 3\Delta_1\Delta_2^2 b_{24}}{\omega_1 + \omega_2},$$

$$\Lambda_{2,10} = \frac{3b_{21} + b_{22}\left(\Delta_1 + 2\Delta_2\right) + b_{23}\left(\Delta_2^2 + 2\Delta_1\Delta_2\right) + 3\Delta_1\Delta_2^2 b_{24}}{3\omega_2 - \omega_1},$$

$$\Lambda_{2,11} = \frac{3b_{21} + b_{22}\left(2\Delta_1 + \Delta_2\right) + b_{23}\left(\Delta_1^2 + 2\Delta_1\Delta_2\right) + 3\Delta_1^2\Delta_2 b_{24}}{2\omega_1},$$

$$\Lambda_{2,12} = \frac{3b_{21} + b_{22}\left(2\Delta_1 + \Delta_2\right) + b_{23}\left(\Delta_1^2 + 2\Delta_1\Delta_2\right) + 3\Delta_1^2\Delta_2 b_{24}}{2\omega_1 + 2\omega_2},$$

$$\Lambda_{2,13} = \frac{3b_{21} + b_{22}\left(\Delta_1 + 2\Delta_2\right) + b_{23}\left(\Delta_2^2 + 2\Delta_1\Delta_2\right) + 3\Delta_1\Delta_2^2 b_{24}}{\omega_1 - \omega_2},$$

$$\Lambda_{2,14} = \frac{3b_{21} + b_{22}\left(\Delta_1 + 2\Delta_2\right) + b_{23}\left(\Delta_2^2 + 2\Delta_1\Delta_2\right) + 3\Delta_1\Delta_2^2 b_{24}}{\omega_1 + 3\omega_2},$$

$$\Lambda_{2,15} = -\Lambda_{2,16} \quad ; \Lambda_{2,16} = \frac{3b_{21} + 3\Delta_2 b_{22} + 3\Delta_2^2 b_{23} + 3\Delta_2^3 b_{24}}{2\omega_2},$$

$$\Lambda_{2,17} = \frac{6b_{21} + b_{22}\left(2\Delta_1 + 4\Delta_2\right) + b_{23}\left(2\Delta_2^2 + 4\Delta_1\Delta_2\right) + 6\Delta_1\Delta_2^2 b_{24}}{\omega_1 + \omega_2},$$

$$\Lambda_{2,18} = \frac{6b_{21} + b_{22}\left(2\Delta_1 + 4\Delta_2\right) + b_{23}\left(2\Delta_2^2 + 4\Delta_1\Delta_2\right) + 6\Delta_1\Delta_2^2 b_{24}}{\omega_2 - \omega_1},$$

$$\Lambda_{2,19} = \frac{6b_{21} + b_{22}\left(2\Delta_1 + 4\Delta_2\right) + b_{23}\left(2\Delta_2^2 + 4\Delta_1\Delta_2\right) + 6\Delta_1\Delta_2^2 b_{24}}{2\omega_2},$$

$$\Lambda_{2,20} = -\Lambda_{2,19}.$$

Appendix D

$$c_{11} = 6b_{11} + \left(4\Delta_2 + 2\Delta_1\right)b_{12} + \left(2\Delta_2^2 + 4\Delta_1\Delta_2\right)b_{13} + 6\Delta_1\Delta_2^2 b_{14},$$

$$c_{12} = 3b_{11} + 3\Delta_1 b_{12} + 3\Delta_1^2 b_{13} + 3\Delta_1^3 b_{14},$$

$$c_{22} = 3b_{21} + 3\Delta_2 b_{22} + 3\Delta_2^2 b_{23} + 3\Delta_2^3 b_{24},$$

$$c_{13} = 3b_{11} + \left(2\Delta_1 + \Delta_2\right)b_{12} + \left(\Delta_1^2 + 2\Delta_1\Delta_2\right)b_{13} + 3\Delta_1^2\Delta_2 b_{14},$$

$$c_{21} = 6b_{21} + \left(4\Delta_1 + 2\Delta_2\right)b_{22} + \left(2\Delta_1^2 + 4\Delta_1\Delta_2\right)b_{23} + 6\Delta_1^2\Delta_2 b_{24}.$$

Appendix E

$$q_1 = a_1\cos(\omega_1 t+\theta_1)+a_2\cos(\omega_2 t+\theta_2)+(\Lambda_{1,1}+\Lambda_{1,2}+\Lambda_{2,1}+\Lambda_{2,2})a_1^3\cos(3\omega_1 t+3\theta_1)$$
$$+(\Lambda_{1,3}+\Lambda_{1,4}+\Lambda_{2,3}+\Lambda_{2,4})a_2^3\cos(3\omega_2 t+3\theta_2)+(\Lambda_{1,5}+\Lambda_{1,8}+\Lambda_{2,5}+\Lambda_{2,8})$$
$$a_1^3\cos(\omega_1 t+\theta_1)+(\Lambda_{1,6}+\Lambda_{1,12}+\Lambda_{2,6}+\Lambda_{2,12})a_1^2 a_2\cos(2\omega_1 t+\omega_2 t+2\theta_1+\theta_2)$$
$$+(\Lambda_{1,6}+\Lambda_{1,12}+\Lambda_{2,6}+\Lambda_{2,12})a_1^2 a_2\cos(2\omega_1 t+\omega_2 t+2\theta_1+\theta_2)$$
$$+(\Lambda_{1,7}+\Lambda_{1,11}+\Lambda_{2,7}+\Lambda_{2,11})a_1^2 a_2\cos(2\omega_1 t-\omega_2 t+2\theta_1-\theta_2) \quad ,$$
$$+(\Lambda_{1,9}+\Lambda_{1,14}+\Lambda_{2,9}+\Lambda_{2,14})a_2^2 a_1\cos(\omega_1 t+2\omega_2 t+\theta_1+2\theta_2)$$
$$+(\Lambda_{1,10}+\Lambda_{1,13}+\Lambda_{2,10}+\Lambda_{2,13})a_2^2 a_1\cos(\omega_1 t-2\omega_2 t+\theta_1-2\theta_2)$$
$$+(\Lambda_{1,15}+\Lambda_{1,16}+\Lambda_{2,15}+\Lambda_{2,16})a_2^3\cos(\omega_2 t+\theta_2)+(\Lambda_{1,17}+\Lambda_{1,18}+\Lambda_{2,17}+\Lambda_{2,18})$$
$$a_1 a_2^2\cos(\omega_1 t+\theta_1)+(\Lambda_{1,19}+\Lambda_{1,20}+\Lambda_{2,19}+\Lambda_{2,20})a_2 a_1^2\cos(\omega_2 t+\theta_2).$$
$$q_2 = \Delta_1 a_1\cos(\omega_1 t+\theta_1)+\Delta_2 a_2\cos(\omega_2 t+\theta_2)+\left(\Delta_1(\Lambda_{1,1}+\Lambda_{1,2})+\Delta_2(\Lambda_{2,1}+\Lambda_{2,2})\right)$$
$$a_1^3\cos(3\omega_1 t+3\theta_1)+\left(\Delta_1(\Lambda_{1,3}+\Lambda_{1,4})+\Delta_2(\Lambda_{2,3}+\Lambda_{2,4})\right)a_2^3\cos(3\omega_2 t+3\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,5}+\Lambda_{1,8})+\Delta_2(\Lambda_{2,5}+\Lambda_{2,8})\right)a_1^3\cos(\omega_1 t+\theta_1)$$
$$+\left(\Delta_1(\Lambda_{1,6}+\Lambda_{1,12})+\Delta_2(\Lambda_{2,6}+\Lambda_{2,12})\right)a_1^2 a_2\cos(2\omega_1 t+\omega_2 t+2\theta_1+\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,6}+\Lambda_{1,12})+\Delta_2(\Lambda_{2,6}+\Lambda_{2,12})\right)a_1^2 a_2\cos(2\omega_1 t+\omega_2 t+2\theta_1+\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,7}+\Lambda_{1,11})+\Delta_2(\Lambda_{2,7}+\Lambda_{2,11})\right)a_1^2 a_2\cos(2\omega_1 t-\omega_2 t+2\theta_1-\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,9}+\Lambda_{1,14})+\Delta_2(\Lambda_{2,9}+\Lambda_{2,14})\right)a_2^2 a_1\cos(\omega_1 t+2\omega_2 t+\theta_1+2\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,10}+\Lambda_{1,13})+\Delta_2(\Lambda_{2,10}+\Lambda_{2,13})\right)a_2^2 a_1\cos(\omega_1 t-2\omega_2 t+\theta_1-2\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,15}+\Lambda_{1,16})+\Delta_2(\Lambda_{2,15}+\Lambda_{2,16})\right)a_2^3\cos(\omega_2 t+\theta_2)$$
$$+\left(\Delta_1(\Lambda_{1,17}+\Lambda_{1,18})+\Delta_2(\Lambda_{2,17}+\Lambda_{2,18})\right)a_1 a_2^2\cos(\omega_1 t+\theta_1)$$
$$+\left(\Delta_1(\Lambda_{1,19}+\Lambda_{1,20})+\Delta_2(\Lambda_{2,19}+\Lambda_{2,20})\right)a_2 a_1^2\cos(\omega_2 t+\theta_2). \quad .$$

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