The Cosserat Point Element (CPE) – A new approach for finite element formulation

M. Jabareen[1]

[1]Department of Structural Engineering and Construction Management, Faculty of Civil and Environmental Engineering, Technion - Israel Institute of Technology, Haifa 32000, Israel

Corresponding author: cvjmah@tx.technion.ac.il

ABSTRACT

The theory of a Cosserat point has been used to develop a 3D brick Cosserat point element (CPE) for the numerical solution of elastic structures that undergo finite deformations. The Cosserat approach postulates an average form of the deformation measure and connects the kinetic quantities to derivatives of a strain energy function. Once this strain energy has been specified, the procedure for obtaining the kinetic quantities needs no integration over the element region and it ensures that the response of the CPE is hyperelastic. Also, the constitutive equations of the CPE were designed to analytically satisfy a nonlinear form of the patch test. Specifically, the strain energy function is additively decomposed into two parts: one controlling the homogeneous deformations and the other controlling the inhomogeneous deformations. Developing a functional form for the strain energy of the inhomogeneous deformations has proven to be challenging. Recently, a functional form of the inhomogeneous strain energy function was developed, which causes the CPE to be a truly user friendly element that can be used with confidence for problems of finite elasticity including: three-dimensional bodies, thin shells and thin rods.

Keywords: Cosserat point element, Enhanced strains elements, Hourglassing, Orthotropic materials, Finite deformations.

Introduction

From the historical point of view, the Cosserat brothers [1] introduced the notion of a deformable body with generalized kinematics in 1909, where the usual vector locating the position of a material point in the body was supplemented by director vectors that characterize additional degrees of freedom. This concept has been used to discuss Cosserat theories that model ”thin” structures which introduce director vectors to characterize material line elements in the dimensions that are reduced due to ”thinness”. Specifically, Naghdi [2] discussed the notion of a Cosserat surface, which is a shell-like structure that is ”thin” in one dimension. Green et al. [3] discussed the notion of a Cosserat curve, which is a rod-like structure that is ”thin” in two dimension and model the cross-section of the rod as deformable. Following this line of thought, it is possible to introduce a model for a point-like structure that is ”thin” in all three-dimensions, like a finite element.

The Cosserat point element (CPE) is a relatively new element technology that is based on the Cosserat point theory [4]. Generally speaking, the CPE theory considers an element as a ”thin” structure in all three dimension and introduces a strain energy function, which characterizes the response of the structure and is suitably restricted so that the element satisfies a nonlinear version of the patch test. The nodal positions are determined by balance laws of director momentum and hyperelastic constitutive equations for intrinsic director couples, which specify the nodal forces. Thus, the nodal forces are related to derivatives of the strain energy function through algebraic relations in a similar manner to the relationship of the stress to derivatives of the strain energy function in the full three-dimensional theory of hyperelastic materials. It was observed in [5, 6] that the three-dimensional brick CPE is a robust, an accurate element that can be used to accurately predict the response of thin plates and shells with only one element through the thickness as well as complicated three-dimensional structures, and it does not exhibit unphysical locking or hourglassing for thin structures or nearly incompressible material response.

In the standard finite element procedure for hyperelastic materials, the response of the element is determined by integrals over the element region which assume that the kinematic approximation is valid point-wise. This formulation may lead to a robust element, but it is known to exhibit unphysical locking for bending dominated structures, and for nearly incompressible materials. Special methods based on enhanced strains, reduced integration with hourglass control and other technologies have been developed to overcome these pathologies. However, it is also known that these improved formulations that are based on enhanced strains can exhibit unphysical hourglassing in regions experiencing combined high compression with bending.

In this paper a generalized three-dimensional brick element for nonlinear orthotropic elastic materials with initially distorted meshes and general orientation of orthotropy is developed using the Cosserat point theory. In particular, in [7] the coefficients of the strain energy function, which controls the response to inhomogeneous deformations, were determined by limiting attention to a rectangular parallelepiped, and it was found that the formulation therein suffers from undesirable sensitivity to initially distorted meshes. Hence, the resulting three-dimensional CPE should have a wider range of applications, such as within realistic three-dimensional engineering problems.

Basic equations of the brick Cosserat point element

Let $\left\{\overline{\mathbf{D}}_i,\ \overline{\mathbf{d}}_i\right\}$ be the nodal position vectors of the brick element in the reference and present configuration, respectively. Then, the location of a material point in both the unstressed reference configuration and the present configuration are, respectively, located by the position vectors $\{\mathbf{X}^*,\ \mathbf{x}^*\}$, which are defined by

$$\mathbf{X}^* = \sum_{j=0}^{7} N^j\left(\theta^i\right)\mathbf{D}_j, \quad \mathbf{D}_j = \sum_{m=0}^{7} A_{jm}\overline{\mathbf{D}}_m, \quad \mathbf{x}^* = \sum_{j=0}^{7} N^j\left(\theta^i\right)\mathbf{d}_j, \quad \mathbf{d}_j = \sum_{m=0}^{7} A_{jm}\overline{\mathbf{d}}_m, \tag{1}$$

where A_{ij} is a transformation matrix given in [7], the directors $\left\{\mathbf{D}_j,\ \mathbf{d}_j\right\}$ represent, respectively, the reference and present element directors, θ^i are convected coordinates, and $N^j\,(j = 0, ..., 7)$ are the element tri-linear shape functions ($N^0 = 1,\ N^1 = \theta^1,\ N^2 = \theta^2,\ N^3 = \theta^3,\ N^4 = \theta^1\theta^2,\ N^5 = \theta^1\theta^3,\ N^6 = \theta^2\theta^3,\ N^7 = \theta^1\theta^2\theta^3$). Next, the point-wise three dimensional deformation gradient reads

$$\mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}^*} = \sum_{i=1}^{3} \frac{\partial \mathbf{x}^*}{\partial \theta^i} \otimes \mathbf{G}^i = \mathbf{F}_0\left(\mathbf{I} + \sum_{i=1}^{3}\sum_{j=1}^{4} N^{j+3}_{,i}\boldsymbol{\beta}_j \otimes \mathbf{G}^i\right), \tag{2}$$

$$\mathbf{F}_0 = \sum_{i=1}^{3} \mathbf{d}_i \otimes \mathbf{D}^i, \quad \boldsymbol{\beta}_i = \mathbf{F}^{-1}\mathbf{d}_{i+3} - \mathbf{D}_{i+3}\ \ (i = 1, ..., 4), \quad b_{3(i-1)+j} = \boldsymbol{\beta}_i \cdot \mathbf{D}_j\ \ (i = 1, 2, 3\ j = 1, 2, 3, 4), \tag{3}$$

where the second order tensor $\mathbf{F}_0$ characterizes the homogeneous deformations and the rigid body rotation, while the four vectors $\boldsymbol{\beta}_i$ characterize the inhomogeneous deformations including six bending modes, three torsional modes and three higher order hourglass modes. It is more convenient to introduce scalars quantities to characterize the inhomogeneous deformations b_i. Unlike the standard Bubnov-Galerkin approach that uses the point-wise deformation gradient (i.e. $\mathbf{F}^*$) for calculating the stresses at the gauss points, the Cosserat approach uses a volume average deformation gradient $\overline{\mathbf{F}}$, which is given by

$$\overline{\mathbf{F}} = \frac{1}{V}\int_{\Omega_0} \mathbf{F}^* d\Omega_0 = \mathbf{F}_0\left(\mathbf{I} + \sum_{j=1}^{4}\boldsymbol{\beta}_j \otimes \mathbf{V}^j\right), \quad \mathbf{V}^j = \frac{1}{V}\sum_{i=1}^{3}\int_{\Omega_0} N^{j+3}_{,i}\mathbf{G}^i d\Omega_0, \quad V = \int_{\Omega_0} d\Omega_0. \tag{4}$$

In addition to postulating average quantities for the deformation, the Cosserat point formulation considers the finite element as a structure with a strain energy function that characterizes its response. This strain energy is additively decomposed into two parts. The first part $m\Sigma^*$ is described by any three-dimensional strain energy function and is associated with the volume average of the right Cauchy-Green deformation tensor $\overline{\mathbf{C}} = \overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}}$. The second part, is associated with inhomogeneous deformation. In particular, the strain energy function of the CPE is

$$m\Sigma = m\Sigma^*\left(\overline{\mathbf{C}}\right) + m\Psi, \quad m\Psi = \frac{V}{2}\sum_{i=1}^{12}\sum_{j=1}^{12} B_{ij}b_ib_j, \quad B_{ji} = B_{ij}, \quad m = \rho_0^* V. \tag{5}$$

The coefficients B_{ij} of the inhomogeneous strain energy function control the accuracy of the CPE for modeling the different inhomogeneous deformation modes and the positive definite of the inhomogeneous strain energy function ensures the stability of the CPE in the sense that there is a proper control on the different inhomogeneous modes. In [8] a new methodology for the determination of the coefficients B_{ij} was presented, which is valid for any elastic material. Now, within the context of the purely mechanical theory, the rate of dissipation of hyperelastic material vanishes and can be written in the following form

$$d^{1/2}\mathsf{D} = d^{1/2}\mathbf{T} : \mathbf{d} + \sum_{m=1}^{4} \mathbf{t}^{m+3} \cdot \left(\mathbf{F}_0\dot{\boldsymbol{\beta}}_m\right) - m\dot{\Sigma} = 0, \quad \mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathrm{T}}\right), \quad \mathbf{l} = \dot{\mathbf{F}}_0\mathbf{F}_0^{-1} = \sum_{i=1}^{3}\dot{\mathbf{d}}_i \otimes \mathbf{d}^i. \tag{6}$$

Assuming that $d^{1/2}\mathbf{T}$ and $\mathbf{t}^i$ $(i = 1, ..., 7)$ are independent of the rates $\{\mathbf{d}, \dot{\beta}_m\}$, it follows that the intrinsic director couples $\mathbf{t}^i$ are

$$\mathbf{t}^{i+3} = d^{1/2}\mathbf{T}\mathbf{v}^i + V\sum_{n=1}^{3}\left(\sum_{j=1}^{12} B_{3(i-1)+n,j}\, b_j\right)\mathbf{d}^n,\ i = 1, .., 4, \quad \mathbf{v}^i = \overline{\mathbf{F}}^{-T}\mathbf{V}^i, \quad \mathbf{t}^i = \left(d^{1/2}\mathbf{T} - \sum_{n=4}^{7}\mathbf{t}^n \otimes \mathbf{d}_n\right)\cdot \mathbf{d}^i,\ i = 1, 2, 3, \qquad (7)$$

and the second order tensor $d^{1/2}\mathbf{T} = 2\overline{\mathbf{F}}\left(m\partial\Sigma^*/\partial\overline{\mathbf{C}}\right)\overline{\mathbf{F}}^{T}$ is related to the volume average of the point-wise Cauchy stress tensor. Finally, the nodal internal forces and the element tangent stiffness, which are needed for the finite element calculations are

$$\mathbf{F}^{\text{int}} = \left\{\bar{\mathbf{t}}^0, \bar{\mathbf{t}}^1, ..., \bar{\mathbf{t}}^7\right\}, \quad \bar{\mathbf{t}}^i = \sum_{j=1}^{7} A_{ji}\mathbf{t}^j, \quad \mathbb{K}_{ij} = \frac{\partial \bar{\mathbf{t}}^i}{\partial \bar{\mathbf{d}}_j}. \qquad (8)$$

Numerical example

In order to demonstrate the robustness of the developed CPE, a plane strain square block which is compressed between two smooth rigid parallel end plates with other two stress free edges is considered. The direction of the material orthotrorpy is given by the vectors $\{\mathbf{M}_1, \mathbf{M}_2\}$ with orientation angle $\theta = 60^o$ (see Fig. 1*a*). Figure 1*b* and *c* show the deformed shapes obtained by the CPE and the enhanced strains element, respectively. It can be seen that the enhanced strains element exhibits an hourglass mode of deformation prior the physical instability point (limit point), while the CPE predicts the physical buckling mode.

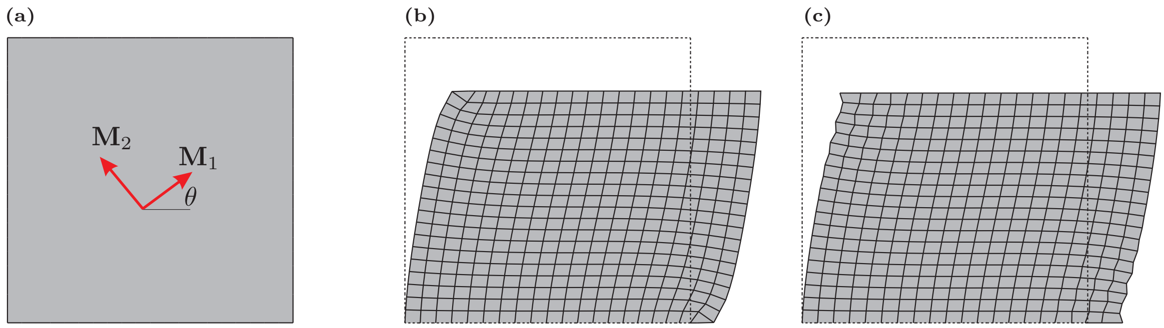

Figure 1. Plane strain compression of a block. (a) Geometry; (b) deformed shape predicated by CPE; (c) deformed shape predicated by enhanced strains element.

References

[1] Cosserat, E., Cosserat, F. (1909) Theorie des corps deformables. Paris. vi+226pp., Chwolson's Traite de Physique, 2nd ed., Paris. Also in translated form, "Theory of deformable bodies", NASA TT F-11, 561 (Washington, D.C.). Clearing house for U.S. Federal Scientific and Technical Information, Springfield, Virginia.

[2] Naghdi, P.M. (1972) The theory of shells and plates. In S. Flugge's Handbuch der Physik, Vol. VIa/2 (ed. Truesdell, C.). *Springer-Verlag, Berlin* 425-640.

[3] Green, A.E., Naghdi, P.M., Wenner, M.L. (1974) On The Theory Of Rods I: Derivations From The Three-dimensional Equations. *Proc. Royal Soc. London A337*, 451-483.

[4] Rubin, M.B. (2000) Cosserat Theories: Shells, Rods and Points., *Solid Mechanics and its Applications*, **79**, Kluwer, The Netherlands.

[5] Loehnert, S., Boerner, E.F.I. , Rubin, M.B., Wriggers, P. (2005) Response of a nonlinear elastic general Cosserat brick element in simulations typically exhibiting locking and hourglassing. *Comput. Mech.* 36:4, 255-265.

[6] Jabareen, M., Rubin. M.B. (2008) A Generalized Cosserat point element (CPE) for isotropic nonlinear elastic materials including irregular 3-D brick and thin structural elements. *Journal of Mechanics of Materials and Structures* **3:8**, 1465-1498.

[7] Jabareen, M., Sharipova, L., Rubin, M.B. (2012) Cosserat point element (CPE) for finite deformation of orthotropic elastic materials. *Comput. Mech.* **49**, 525-54.

[8] Jabareen, M., Mtanes, E. (2015) A 3D Cosserat point element (CPE) for nonlinear orthotropic solids: Generalization for an initially distorted mesh and an arbitrary orientation of material orthotropy. *Accepted for publication in Int. J. Numer. Meth. Engng.* **DOI: 10.1002/nme.5000**.