Multi-patches based B-Spline method

for Solid and Structure

Yanan Liu* Bin Hu

China Special Equipment Inspection and Research Institute，Beijing 100029, China.

*Presenting author: liuyanan@csei.org.cn
*Corresponding author: liuyanan@csei.org.cn

Abstract

In this paper, the solution domain is divided into multi-patches on which B-Spline basis functions are used for approximation. The different B-Spline patches are connected by a transition region which is described by several elements. The basis functions in different B-Spline patches are modified in the transition region to ensure the basic polynomial reconstruction condition and the compatibility of displacements and their gradients. This new method is applied to the stress analysis of 2D elasticity problems in order to investigate its performance. Numerical results show that the present method is accurate and stable.

Keywords: B-Spline patches, Transition region, B-Spline basis functions.

Introduction

B-spline functions have been widely used in numerical analysis and simulation for decades. In fact, a considerable body of literature now exists on the application of uniform and non-uniform B spline techniques to the solution of partial differential equations (PDEs) and mechanics problems. The recent studies of B-spline method can be found in some articles [1]-[7]. The B-spline basis functions have compact support and lead to banded stiffness matrices. They can be used to construct piecewise approximations that provide higher order of continuity depending on the order of the polynomial basis. The B-spline basis functions form a partition of unity, which is an important property for convergence of the approximate solutions. As they are polynomials, accurate integration can be performed by using the Gauss quadrature. The B-spline approximation has good reproducing properties; thus, it is able to represent constant strains exactly. Compared to orthogonal or biorthogonal wavelets scaling functions and the shape functions constructed by meshless method, B spline functions are more simple and easy to work with for numerical analysis.

The main disadvantage of the general B-spline-based methods is that the scale used in approximation is usually uniform. In order to effectively simulate the local complicated deformation, the scale used in approximation should be very small. In this case, the computational efficiency will be very low. So it is desirable that the scales used for function approximation in solution domain are different. A more general approach that uses non-uniform rational B-splines (NURBS) [8]-[11] for the analysis has been developed. The method is referred to as the isogeometric analysis method because the geometry is also represented using NURBS basis functions to get an exact geometric representation. This method can achieve the traditional *h*- and *p*-adaptive refinement as well as *k*-refinement and get better solutions due to the superior basis functions. However, it is necessary to generate meshes that conform to the geometry of the analysis in this method.

In this paper, the solution domain is divided into multi-patches. The B-spline basis functions are directly used to approximate the unknown field functions in each patch. Thus, generation of conforming mesh is avoided in this approach. Different scales can be used in approximation for corresponding patch. The fine scale is used for function approximation in the patches where the deformation is complicated. The coarse scales are used for approximation in other patches where the deformation is relatively simple. A transition domain is used for combination of different B-spline patches. An algorithm is developed to modify the B-spline basis functions in the transition domain. The compatibility conditions on the interface between the different patches can be satisfied. The numerical examples of 2-D elasticity analysis are given to illustrate the stability and the effectiveness of the present method.

2. Approximation of 2-D functions by B-spline with single scale

The m degree B-spline is defined as

$$N_m(x) = N_{m-1} * N_1 = \int_0^1 N_{m-1}(x-t)dt, \ \ m \geq 2 \tag{1}$$

where

$$N_1(x) = \begin{cases} 1, x \in [0,1) \\ 0, \quad\quad \text{else} \end{cases} \tag{2}$$

The major properties of B-spline are

$$\begin{gathered} \text{Supp} N_m = [0, m] \\ N_m{}'(x) = N_{m-1}(x) - N_{m-1}(x-1) \\ N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1) \end{gathered} \tag{3}$$

B-spline functions can be used as basis functions to approximate the function u defined on interval $[a,b]$.

$$u(x) = \sum_k c_k N_m^{i,k}(x) \tag{4}$$

where, $N_m^{i,k}(x) = N_m(1/i \cdot x - k)$ and i denotes the scale in approximation. According to properties of B-spline, the support of $N_m^{i,k}(x)$ is

$$\text{Supp} N_m^{i,k} = [ik, i(m+k)] \tag{5}$$

In approximation Eq.(4), the B-spline functions $N_m^{i,k}(x)$ should satisfy the following condition

$$\text{Supp} N_m^{i,k} \cap [a,b] \neq \varnothing \tag{6}$$

The basis functions for the higher-dimensional problems are constructed by taking the product of the basis functions for 1-D B-spline. In this case, the approximation of 2-D function $u(x, y)$ by B-spline function can be expressed as

$$u(x, y) = \sum_{k,l} c_{k,l} N_m^{i,k}(x) N_m^{j,l}(y) \tag{7}$$

where, $N_m^{i,k}(x) N_m^{j,l}(y)$ are 2-D B-spline basis functions, i and j are respectively the scales of x direction and y direction in approximation. For 2-D problems in general domains Ω, the 2-D B-spline basis functions which meet the following condition should be used in function approximation.

$$\text{Supp} N_m^{i,k}(x) N_m^{j,l}(y) \cap \Omega \neq 0 \tag{8}$$

Similar to finite element method and meshless methods, the approximation equation can be written as

$$u(x, y) = \sum_{h=1}^{N} \phi_h(x, y) c_h \tag{9}$$

where, $\phi_h(x, y) = N_m^{i,k}(x) N_m^{j,l}(y)$ is similar to shape functions in finite element method and meshless methods, c_h are the generalized displacement related to $\phi_h(x, y)$ and N is the number of 2-D B-spline functions used in approximation.

3. Coupling of different B-Spline patches

3.1 Basic formulations

The equations for the elastic problem are expressed as follows

$$
\begin{aligned}
&\sigma_{ij,j} + b_i = 0 && \text{in} \quad \Omega \\
&\sigma_{ij} n_j = \bar{t}_i && \text{on} \quad \Gamma_{\mathrm{t}} \\
&u_i = \bar{u}_i && \text{on} \quad \Gamma_u
\end{aligned}
\tag{10}
$$

where σ_{ij} is the stress tensor, b_i is the body force, $\bar{t}_i$ and $\bar{u}_i$ are respectively the prescribed boundary tension and displacement, and n_j is the unit outward normal to domain Ω. Consider the virtual displacement principle

$$\int_\Omega (\sigma_{ij,j} + b_i)\delta u_i d\Omega + \int_{\Gamma_t} (\sigma_{ij} n_j - \bar{t}_i)\delta u_i d\Gamma = 0 \tag{11}$$

where δu_i is the variation of real displacement. From Eq. (11), the weak form is

$$\int_\Omega \delta e_{ij} \sigma_{ij} d\Omega = -\int_{\Gamma_t} \delta u_i \bar{t}_i d\Gamma + \int_\Omega \delta u_i b_i d\Omega \tag{12}$$

where δu_i vanishes and $u_i = \bar{u}_i$ on Γ_u.

Consider a division with two patches and a transition region in a given region Ω as shown in Figure 1. In 2-D problem, the approximation for the displacement field u and v can be respectively written as

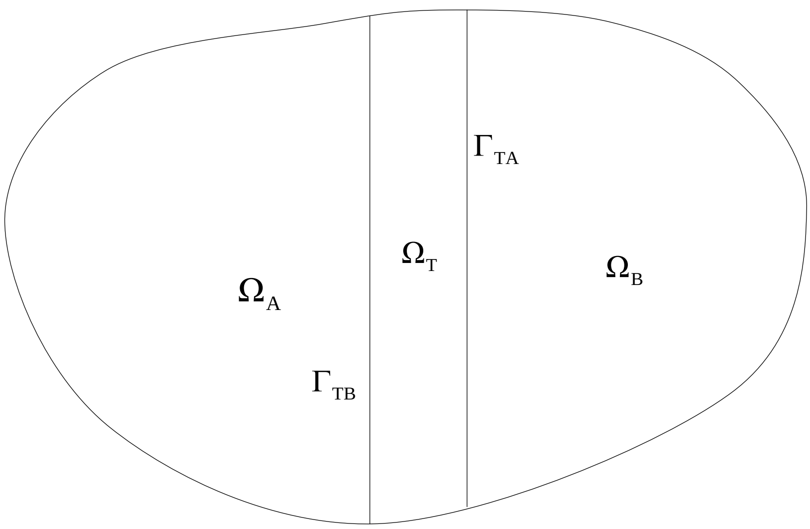

Figure 1. The problem domain is divided into two patches and a transition region

$$\begin{cases} u_{\mathrm{A}}(x,y) = \sum_{i=1}^{n_{\mathrm{W}}} a_i^{\mathrm{A}u} \phi_i^{\mathrm{A}}(x,y) \\ v_{\mathrm{A}}(x,y) = \sum_{i=1}^{n_{\mathrm{A}}} a_i^{\mathrm{A}v} \phi_i^{\mathrm{A}}(x,y) \end{cases} \qquad (x,y) \in \Omega_{\mathrm{A}} + \Omega_{\mathrm{T}} \tag{13}$$

$$\begin{cases} u_{\mathrm{B}}(x,y) = \sum_{i=1}^{n_{\mathrm{F}}} a_i^{\mathrm{B}u} \phi_i^{\mathrm{B}}(x,y) \\ v_{\mathrm{B}}(x,y) = \sum_{i=1}^{n_{\mathrm{B}}} a_i^{\mathrm{B}v} \phi_i^{\mathrm{B}}(x,y) \end{cases} \qquad (x,y) \in \Omega_{\mathrm{B}} + \Omega_{\mathrm{T}} \tag{14}$$

where, n_{A} and n_{B} is respectively the number of basis functions used in approximation. It is obvious that the two kinds of approximation functions are not compatible and should be modified in the transition region.

3.2 Transition region and modified basis functions

In 2D problems, the transition region can be described by several elements.

$$\begin{cases} x_k(\xi,\eta) = \sum_{i=1}^{n} N_i^{\mathrm{F}}(\eta,\xi) x_i^k \\ y_k(\xi,\eta) = \sum_{i=1}^{n} N_i^{\mathrm{F}}(\eta,\xi) y_i^k \end{cases} \qquad -1 \le \eta \le 1, -1 \le \xi \le 1, 1 \le k \le n_{\mathrm{F}} \tag{15}$$

where, N_i^{F} is the shape function of four nodes plain element and n_{F} is the number of element. The basis functions should be modified in transition region. A weight function based on the transition region should be introduced into modification. The modified basis functions in the transition region can be expressed as

$$\begin{cases} \phi_{k,n}^{\mathrm{Am}}(\xi,\eta) = \phi_n^{\mathrm{A}}(x(\xi,\eta), y(\xi,\eta)) * w(\eta) \\ \phi_{k,n}^{\mathrm{Bm}}(\xi,\eta) = \phi_n^{\mathrm{B}}(x(\xi,\eta), y(\xi,\eta)) * (1 - w(\eta)) \end{cases} \qquad (x,y) \in \Omega_{\mathrm{T}} \tag{16}$$

The following functions can be chosen as weight function

$$w(\eta) = 1 - 6 * (\frac{\eta+1}{2})^2 + 8 * (\frac{\eta+1}{2})^3 - 3 * (\frac{\eta+1}{2})^4 \qquad -1 \le \eta \le 1 \tag{17}$$

Then, the approximation in transition region can be expressed as

$$\begin{cases} u_k(\xi,\eta) = \sum_i a_{\mathrm{m},i}^{\mathrm{A}u} \phi_{k,i}^{\mathrm{Am}}(\xi,\eta) + \sum_j a_{\mathrm{m},j}^{\mathrm{B}u} \phi_{k,j}^{\mathrm{Bm}}(\xi,\eta) \\ v_k(\xi,\eta) = \sum_i a_{\mathrm{m},i}^{\mathrm{A}v} \phi_{k,i}^{\mathrm{Am}}(\xi,\eta) + \sum_j a_{\mathrm{m},j}^{\mathrm{B}v} \phi_{k,j}^{\mathrm{Bm}}(\xi,\eta) \end{cases} \qquad -1 \le \eta \le 1, -1 \le \xi \le 1, 1 \le k \le n_{\mathrm{F}} \tag{18}$$

Then, the approximation formula (13) and (14) should be rewritten as

$$\begin{cases} u_{\mathrm{A}}(x,y) = \sum_{i=1}^{n_{\mathrm{A}}} a_i^{\mathrm{A}u} \phi_i^{\mathrm{A}}(x,y) \\ v_{\mathrm{A}}(x,y) = \sum_{i=1}^{n_{\mathrm{A}}} a_i^{\mathrm{A}v} \phi_i^{\mathrm{A}}(x,y) \end{cases} \qquad (x,y) \in \Omega_{\mathrm{A}} \tag{19}$$

$$\begin{cases} u_{\mathrm{B}}(x,y) = \sum_{i=1}^{n_{\mathrm{B}}} a_i^{\mathrm{B}u} \phi_i^{\mathrm{B}}(x,y) \\ v_{\mathrm{B}}(x,y) = \sum_{i=1}^{n_{\mathrm{B}}} a_i^{\mathrm{B}v} N_i^{\mathrm{B}}(x,y) \end{cases} \qquad (x,y) \in \Omega_{\mathrm{B}} \tag{20}$$

Eventually, a group of linear algebraic equations can be obtained by introducing the approximations formula (18), (19) and (20) into weak form (12).

$$\mathbf{K}\mathbf{a} = \mathbf{f} \tag{21}$$

where,

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{\mathrm{A}} & \mathbf{K}_{\mathrm{AB}} \\ \mathbf{K}_{\mathrm{AB}}^{\mathrm{T}} & \mathbf{K}_{\mathrm{B}} \end{bmatrix} \tag{22}$$

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_{\mathrm{A}} \\ \mathbf{a}_{\mathrm{B}} \end{bmatrix} \tag{23}$$

$$\mathbf{f} = \begin{bmatrix} f_{\mathrm{A}} \\ f_{\mathrm{B}} \end{bmatrix} \tag{24}$$

and,

$$\mathbf{a}_{\mathrm{A}} = [a_1^{\mathrm{A}u}, a_1^{\mathrm{A}v}, \cdots, a_{n_{\mathrm{A}}}^{\mathrm{A}u}, a_{n_{\mathrm{A}}}^{\mathrm{A}v}]^{\mathrm{T}} \tag{25}$$

$$\mathbf{a}_{\mathrm{B}} = [a_1^{\mathrm{B}u}, a_1^{\mathrm{B}v}, \cdots, a_{n_{\mathrm{B}}}^{\mathrm{B}u}, a_{n_{\mathrm{B}}}^{\mathrm{B}v}]^{\mathrm{T}} \tag{26}$$

$$\mathbf{K}_{\mathrm{A}} = \int_{\Omega_{\mathrm{A}}} \mathbf{B}_{\mathrm{A}}{}^{\mathrm{T}} \mathrm{D} \mathbf{B}_{\mathrm{A}} d\Omega + \int_{\Omega_{\mathrm{T}}} \mathbf{B}_{\mathrm{A}}{}^{\mathrm{T}} \mathrm{D} \mathbf{B}_{\mathrm{A}} d\Omega \tag{27}$$

$$\mathbf{K}_{\mathrm{B}} = \int_{\Omega_{\mathrm{B}}} \mathbf{B}_{\mathrm{B}}{}^{\mathrm{T}} \mathbf{D} \mathbf{B}_{\mathrm{B}} d\Omega + \int_{\Omega_{\mathrm{T}}} \mathbf{B}_{\mathrm{B}}{}^{\mathrm{T}} \mathbf{D} \mathbf{B}_{\mathrm{B}} d\Omega \tag{28}$$

$$\mathbf{K}_{\mathrm{AB}} = \int_{\Omega_{\mathrm{T}}} \mathbf{B}_{\mathrm{A}}{}^{\mathrm{T}} \mathrm{D} \mathbf{B}_{\mathrm{B}} d\Omega \tag{29}$$

$$\mathbf{f}_{\mathrm{A}}{}^{\mathrm{T}} = \int_{\Omega_{\mathrm{A}}} \boldsymbol{\varphi}^{\mathrm{T}} \mathbf{b} d\Omega + \int_{\Omega_{\mathrm{T}}} \boldsymbol{\varphi}^{\mathrm{T}} \mathbf{b} d\Omega + \int_{\Gamma_{\mathrm{A}}} \boldsymbol{\varphi}^{\mathrm{T}} \bar{\mathbf{t}} d\Gamma \tag{30}$$

$$\mathbf{f}_{\mathrm{B}}{}^{\mathrm{T}} = \int_{\Omega_{\mathrm{B}}} \mathbf{N}^{\mathrm{T}} \mathbf{b} d\Omega + \int_{\Omega_{\mathrm{T}}} \mathbf{N}^{\mathrm{T}} \mathbf{b} d\Omega + \int_{\Gamma_{\mathrm{B}}} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{t}} d\Gamma \tag{31}$$

$\mathbf{D}$ is the 2-D elasticity matrix.

$$\mathbf{D} = \frac{E_0}{(1-\nu_0^2)} \begin{bmatrix} 1 & \nu_0 & 0 \\ \nu_0 & 1 & 0 \\ 0 & 0 & \frac{1-\nu_0}{2} \end{bmatrix}$$

Plain stress

$$E_0 = E, \quad \nu_0 = \nu$$

Plain strain

$$E_0 = \frac{E}{1-\nu^2}, \quad \nu_0 = \frac{\nu}{1-\nu}$$

$$\mathbf{B}_{\mathrm{A}} = \mathbf{L}\boldsymbol{\phi}^{\mathrm{A}}$$

$$\mathbf{B}_{\mathrm{B}} = \mathbf{L}\boldsymbol{\phi}^{\mathrm{B}}$$

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\boldsymbol{\phi}^{\mathrm{A}} = \begin{bmatrix} \phi_1 & 0 & \cdots & \phi_{n_{\mathrm{A}}} & 0 \\ 0 & \phi_1 & \cdots & 0 & \phi_{n_{\mathrm{A}}} \end{bmatrix}$$

$$\boldsymbol{\phi}^{\mathrm{B}} = \begin{bmatrix} \phi_1 & 0 & \cdots & \phi_{n_{\mathrm{B}}} & 0 \\ 0 & \phi_1 & \cdots & 0 & \phi_{n_{\mathrm{B}}} \end{bmatrix}$$

4 Numerical examples

In this part, numerical simulation of some 2-D plain elasticity problems is presented using the present method. The results are compared with those calculated by finite element method or analytical results to show the validity of the proposed method. For simplification, the units are omitted in this paper.

Cantilever beam

A cantilever beam is analyzed by the presented method. As shown in Figure 2, the beam has a dimension of length $L=10$ and height $h=2$ and is subject to a parabolic traction with $P=-300$ and $p_y=-0.75P(1-(y-1)^2)$. The beam has a unit thickness and a plane strain problem is considered. The Young's modulus is set to $E=2.1\times10^4$ and Poisson's ration is set to $\nu=0.49$. In this problem, the analytical results of stress are expressed as follows

$$\sigma_x=\frac{P}{I}(L-x)(y-1)\ ,\quad \sigma_y=0\ ,\quad \sigma_{xy}=-\frac{P}{2I}\left[(\frac{h}{2})^2-(y-1)^2\right]$$

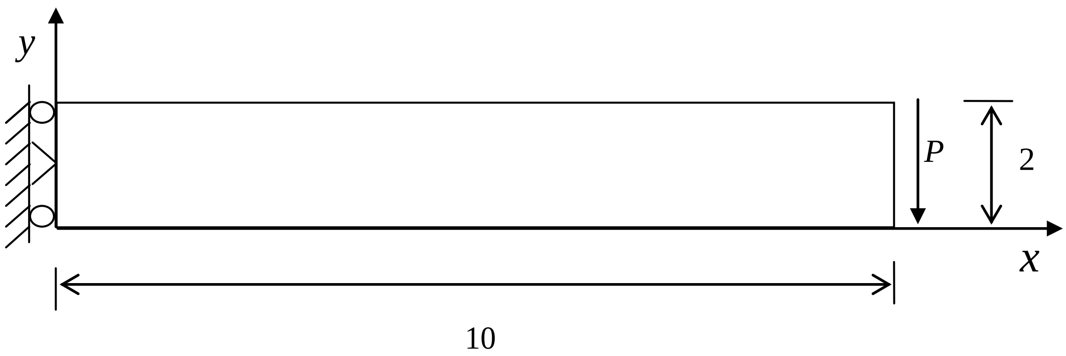

Figure 2. Cantilever beam under a parabolic traction at the free end

The problem domain is divided into two patches and a transition region as shown in Figure 3. Cubic B-Spline is used in this simulation. The scales used for approximation in two patches are denoted by $\mathrm{A}_x, \mathrm{A}_y$ and $\mathrm{B}_x, \mathrm{B}_y$, respectively. The width of transition region is expressed by t and the Γ_{TA} is fixed at $x=4$. The different parameters related to

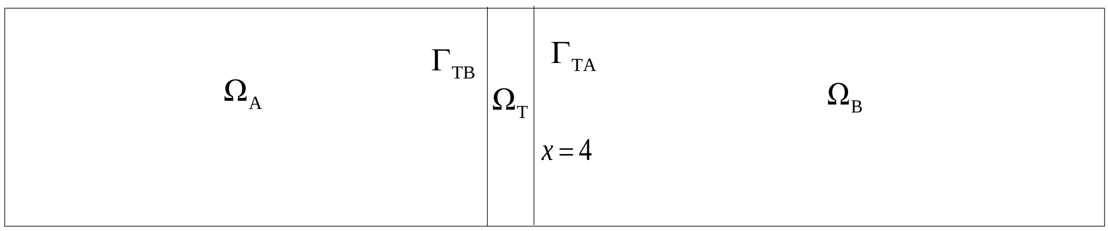

Figure 3. The patches in 2D beam problem

transition region are studied in this simulation. Figure 4 shows the results of σ_{xy} along $x=3$ and $x=5$ with $t=0.2$. It can be found that the results computed by the present method agree well with the analytical results.

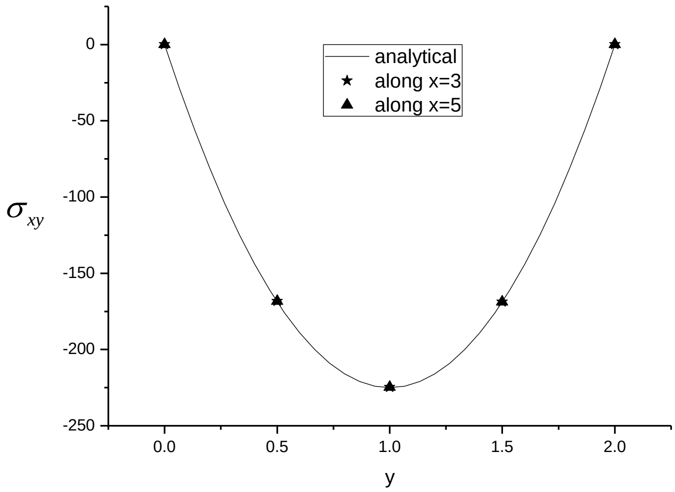

Figure 4. The comparison of shear stress

Conclusions

In this paper, the B-spline basis functions are directly used to approximate the unknown field functions in multi-patches. The generation of conforming mesh is avoided in this approach. Different scales are used in approximation for corresponding patch. A transition domain is used for combination of different B-spline patches. The B-spline basis functions are modified to satisfy the high-order compatibility conditions on the interface between the different patches. The computational efficiency of this method is much higher than single patch based single scale approach. Numerical examples for 2-D elasticity problems illustrate that this B-spline method is effective and stable for solving elasticity problems.

References

[1] Naginoa, H., Mikamia, T. and Mizusawa, T. (2008) Three-dimensional free vibration analysis of isotropic rectangular plates using the B-spline Ritz method, *Journal of Sound and Vibration* 317, 329–353.
[2] Caglar, N. and Caglar, H. (2009) B-spline method for solving linear system of second-order boundary value problems, *Computers Mathematics with Applications* 57, 757-762.
[3] Kagan, P., Fischer, A. and Bar-Yoseph, P.Z. (2003) Mechanically based models: Adaptive refinement for B-spline finite element, *International Journal for Numerical Methods in Engineering* 57, 1145–1175.
[4] Burla, R.K. and Kumar, A.V. (2008) Implicit boundary method for analysis using uniform B-spline basis and structured grid, *International Journal for Numerical Methods in Engineering* 76, 1993–2028.
[5] Lakestani, M. and Dehghan, M. (2009) Numerical Solution of Fokker—Planck Equation Using the Cubic B-Spline Scaling Functions, *Numerical Methods for Partial Differential Equations* 25, 418–429.
[6] Liu, Y., Sun, L., Xu, F., Liu, Y. and Cen, Z. (2011) B spline-based method for 2-D large deformation analysis, *Engineering Analysis with Boundary Elements* 35, 761-767.
[7] Liu, Y., Sun, L., Liu, Y. and Cen, Z. (2011) Multi-scale B-spline method for 2-D elastic problems, *Applied Mathematical Modelling* 35, 3685-3697.
[8] Hughes, T.J.R., Cottrell, J.A. and Bazilevs, Y. (2005) Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, *Computer Methods in Applied Mechanics and Engineering* 194, 4135–4195.
[9] Cottrell, J.A., Reali, A., Bazilevs, Y. and Hughes, T.J.R. (2006) Isogeometric analysis of structural vibrations, *Computer Methods in Applied Mechanics and Engineering* 195, 5257–5296.
[10] Cottrell, J.A., Hughes, T.J.R. and Reali, A. (2007) Studies of refinement and continuity in isogeometric analysis, *Computer Methods in Applied Mechanics and Engineering* 196, 4160–4183.
[11] Bazilevs, Y., Calo, V.M., Hughes, T.J.R. and Zhang, Y. (2008) Isogeometric fluid–structure interaction: theory, algorithms, and computations, *Computational Mechanics* 43, 3–37.