Analysis of the first modal shape using case studies

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Abstract

Eigenvector analysis can be performed to determine the shapes and frequencies of the undampened free vibration modes of a system. These natural modes provide excellent insight into the behavior of a particular structure. Eigen vector analysis involves solving the generalized eigenvalue problem, which considers the stiffness and mass matrix of a structure. When a geometric nonlinear study must be performed, a situation that commonly occurs in the analysis of slender structures, nonlinear analysis or a more complete and rigorous evaluation that considers both parts of the total matrix is required. For instance, slender structures possess a small first frequency of vibration, less than 1 Hz, and can resonate due to wind excitation. The first frequency and shape of vibration are the most important parameters for calculating the response of a structure to environmental excitation. Therefore, when modal analysis depends on the stiffness of the structure, the effect of a reduction in stiffness on the modal shape of vibration must be determined. To this end, case studies were evaluated using the finite element method (FEM), considering and neglecting the geometric portion of the stiffness matrix. Mathematic functions were also applied for comparison.

Keywords: Modal Shape, Geometric Stiffness, Nonlinear Analysis, Computational Simulation, Mathematic functions, Case Studies

Introduction

For structures with a first natural frequency less than 1 Hz, the dynamic effects of wind are too important to be considered as pure static effort or deterministic in nature, which would only provide a rough approximation. Regarding the importance of the dynamic effects of wind, Durbey and Hansen (1996) suggested that flexible structures vibrate in different modes, frequencies and shapes when excited by the wind. Further, they stated that the dynamic effect of wind may allow slender structures to display resonance.

In many countries, models that consider the effects of wind in design structures are provided by governing codes. Many of these models consider that average wind speeds produce a static effect, whereas fluctuations or gusts of wind produce important oscillations, especially in tall constructions. When dealing with the dynamic response to the average wind speed, fluctuations are considered to occur in the band of the lower frequencies of the structure. This model of dynamic analysis was also considered by Simiu and Scalan (1996), who suggested that induced vibration analysis for floating loads was a necessary model component. Moreover, constructions with a basic period greater than 1 s and frequencies up to 1 Hz can undergo a floating response in the direction of the wind. Although the frequencies and vibration shapes of a structure should be considered, the most important parameter is the fundamental frequency.

Modal analysis and vibration shapes

A classical method for the dynamic analysis of a structure is modal analysis, in which sufficient information on the system or structure is obtained to reproduce their dynamics. Carrion et al. (2014) previously indicated that the natural frequencies (eigenvalues) and modes of vibration (eigenvectors) of the system are relevant information for classical modal analysis. Carrion further stated that a well-known concept used in the finite element method (FEM) is the stiffness matrix, which is used to relate the external forces applied at the nodes of the structural element to the nodal displacement.

Structural dynamics can be employed to obtain solutions to homogeneous differential equations, the shape of which represents vibration modes that exist in the coordinate system at the same frequency range and occur harmonically in time. The equation describes the vibration of the system according to a normal mode of vibration and corresponds to the frequency. After deriving the solution twice with respect to time and canceling the harmonic function, the homogeneous algebraic equations shown in Eq. (1) were obtained. In the equation, ω^2 are the eigenvalues, and Φ are the eigenvectors in the FEM environment.

$$\left[\left[K\right]-\omega^2\left[M\right]\right]\ \left\{\Phi\right\} = 0 \tag{1}$$

[K] is the total stiffness matrix, which is composed of two parts, one being conventional, as shown in Eq. (2), the other being geometric, as shown in Eq. (3). [M] is the known mass matrix, pertaining to modal analysis with geometric nonlinear characteristics. When the mass matrix is a discrete mass distribution (lumped mass) of the structural system, a diagonal matrix containing the masses and moments of inertia for the nodal displacements is obtained.

$$\left[k_0\right] = E\begin{bmatrix} \frac{A}{L} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ & \frac{12I}{L^3} & \frac{6I}{L^2} & 0 & -\frac{12I}{L^3} & \frac{6I}{L^2} \\ & & \frac{4I}{L} & 0 & -\frac{6I}{L^2} & \frac{2I}{L} \\ & & & \frac{A}{L} & 0 & 0 \\ & symmetric & & & \frac{12I}{L^3} & -\frac{6I}{L^2} \\ & & & & & \frac{4I}{L} \end{bmatrix} \tag{2}$$

$$\left[k_g\right] = \frac{F}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ & & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ & & & 0 & 0 & 0 \\ & symmetric & & & \frac{6}{5} & -\frac{L}{10} \\ & & & & & \frac{2L^2}{15} \end{bmatrix}. \tag{3}$$

The mathematic solution to the dynamic problem is a polynomial equation of degree n that contains the variable ω^2 and is commonly known as the frequency equation. The n solutions for ω_I are real and positive and are considered the natural frequencies of the system. The smallest frequency is typically denoted as ω_1, while the largest frequency is denoted as ω_n. Thus, n modes of vibration can be determined and collected in a modal $n \times n$ matrix, which contains columns representing the n modes of undampened, normalized free vibration (Brazil, 2004). Each pair of eigenvalues and eigenvectors corresponds to a frequency and mode of vibration for the system. To consider values and characteristic vectors equal in number to the nodal displacements of the system, Venancio Filho (1975) suggested that Eq. (1) can be written as follows:

$$\left[\Phi\right]\left[\omega^2\right] = \left[K\right]\left[M\right]^{-1}\left[\Phi\right] \tag{4}$$

where [ω^2] is the diagonal matrix of order n and consists of the natural frequencies squared, and [Φ] is an $n \times n$ matrix and contains columns corresponding to the normal modes of vibration. The term [K][M]$^{-1}$ is a dynamic matrix, as previously mentioned by Blessmann (2005).

The formulation corresponding to the previous exposition of the FEM is a geometric nonlinear formulation and is based on the geometric stiffness matrix. Geometric stiffness has been introduced in several analyses of the FEM when nonlinear effects or geometric nonlinearity (GNL) are considered. The interpolation functions normally used in FEM formulations to determine the full stiffness matrix are third-degree polynomials, such as those evaluated by Filho (1975) and Wilson and Bathe (1976).

Computer models of actual structures were developed in the present study using a FEM-based computer modeling program, and modal analysis was performed linearly and nonlinearly to obtain the shape of the first mode of vibration. For comparative purposes, mathematic functions, such as the trigonometric function given in Eq. (5), the polynomial function given in Eq. (6), and the potential function given in Eq.(7). All of the functions were considered to be valid throughout the entire domain of the structure.

Trigonometric function

$$\phi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right). \tag{5}$$

Polynomial function

$$\theta(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}. \tag{6}$$

Potential function

$$\psi(x) = \left(\frac{x}{L}\right)^{\gamma}. \tag{7}$$

The value of γ was determined in the present research.

Analysis of the first modal shape using case studies

Extremely slender structures possessing frequencies of the first vibration mode less than 1 Hz were selected. Modal analysis was achieved using finite element models, according to SAP2000 (integrated software for structural analysis and design, Analysis Reference Manual, Computer and Structures, Inc., Berkeley, California, USA), a commercial software package. Modal shapes for the structures were obtained linearly and nonlinearly. The procedure used to calculate the nonlinear modal shape considered geometric stiffness; therefore, the influence of axial loads was inserted in the stiffness matrix. The structures were modeled using bar elements with constant and variable cross sections, as appropriate.

Structure with a slenderness index of 310

The evaluated structure was 48 m high and possessed a hollow circular section with a variable external diameter (ϕ_{ext}) and thickness (t). The slenderness index of the pole was set to 310. The geometric details are shown in Figure 1(b), where t is the thickness of the wall of each segment of the structure. The metal pole was used to support the transmission system for mobile telephone signals. Table 1 lists the structural parameters and existing devices on the structure, and Table 2 specifies the structural properties and model discretization values.

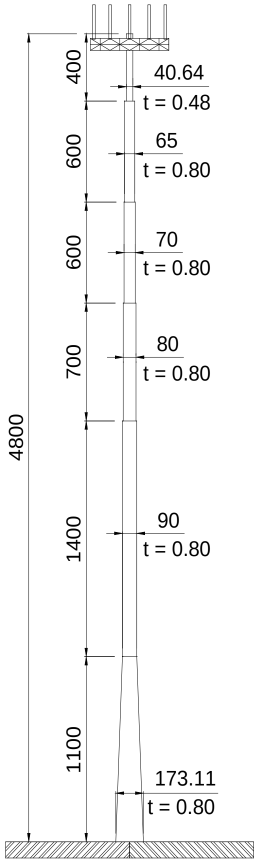
Table 1. Devices and weights on the structure

Device	Height	Weight and distributed weight
Pole	from 0 to 48 m	7850 kN m^{-3}
Ladder	from 0 to 48 m	0.15 kN m^{-1}
Cables	from 0 to 48 m	0.25 kN m^{-1}
Antenna and supports	48 m	3.36 kN

Table 2. Structural properties and discretization of the FEM model

Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)
48.00	40.64	0.48	30.00	80.00	0.80	20.00	90.00	0.80	10.00	97.56	0.80
46.00	40.64	0.48	29.00	80.00	0.80	19.00	90.00	0.80	9.00	105.11	0.80
44.00	40.64	0.48	28.00	80.00	0.80	18.00	90.00	0.80	8.00	112.67	0.80
42.00	65.00	0.80	27.00	80.00	0.80	17.00	90.00	0.80	7.00	120.22	0.80
40.00	65.00	0.80	26.00	80.00	0.80	16.00	90.00	0.80	6.00	127.78	0.80
38.00	65.00	0.80	25.00	80.00	0.80	15.00	90.00	0.80	5.00	135.33	0.80
36.00	70.00	0.80	24.00	90.00	0.80	14.00	90.00	0.80	4.00	142.89	0.80
34.00	70.00	0.80	23.00	90.00	0.80	13.00	90.00	0.80	3.00	150.44	0.80
32.00	70.00	0.80	22.00	90.00	0.80	12.00	90.00	0.80	2.00	158.00	0.80
31.00	80.00	0.80	21.00	90.00	0.80	11.00	90.00	0.80	1.00	165.56	0.80
									0.00	173.11	0.80

(a) Slender metallic pole



(b) Geometric details

Figure 1. Slender metallic pole and its geometric details

The modal shapes obtained by FEM and the aforementioned mathematic functions are provided in the graph shown in Figure 3. The exponent of the potential function that best fit the curve was equal to 1.965.

Structure with a slenderness index of 256

This investigated structure is a truncated cone metallic pole with 52 cm and 82 cm top and bottom diameters respectively. It is intended for the sustaining of the mobile phone broadcasting system. It is 30 meters high, hollow section. The external diameter (ϕ_{ext}) and thickness (t) vary along of the height. The assessed slenderness of the structure is 256.

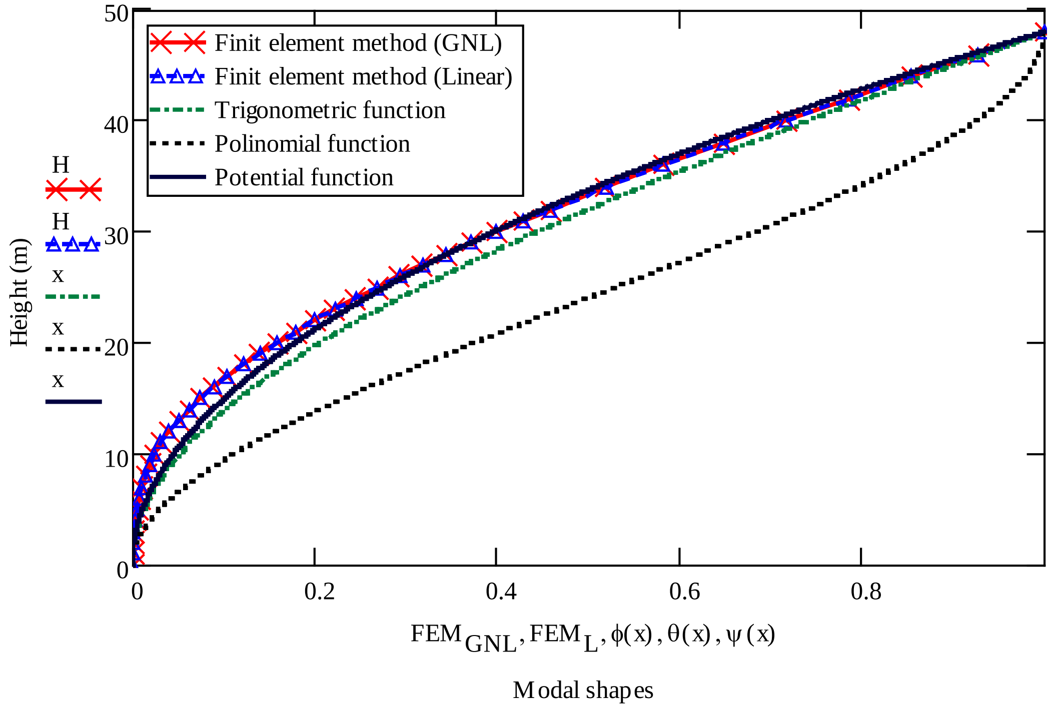


Figure 2. Modal shapes of the structure with slenderness 310

The structure data were acquired in the field. The diameters were measured with a metallic tape measure and the thickness with ultrasound equipment. For a given vertical line, several thickness measurements were carried out to obtain a relative average of the band. The union of the pole segments is formed by successive fittings, by placing and screw-fastening the metallic parts. Each superpositioning band has 20 cm length. In these joint areas, the thickness of the transverse section corresponds to the sum of the measures of the superpositioning bands, conform is indicated in Figure 3. In Table 3 it can be found the properties and the discretization used to model the structure.

Table 3: Structural properties and discretization of the FEM model.

Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)
30.00	52.00	0.60	20.00	62.00	0.60	10.00	72.00	0.76
29.00	53.00	0.60	19.00	63.00	0,60	9.00	73.00	0.76
28.00	54.00	0.60	18.10	63.90	0.60	8.00	74.00	0.76
27.00	55.00	0.60	17.90	64.10	0.60	7.00	75.00	0.76
26.00	56.00	0,60	17.00	65.00	0.60	6.10	75.90	0.76
25.00	57.00	0.60	16.00	66.00	0.60	5.90	76.10	0.76
24.10	57.90	0.60	15.00	67.00	0.60	5.00	77.00	0.76
23.90	58.10	0.60	14.00	68.00	0.60	4.00	78.00	0.76
23.00	59.00	0.60	13.00	69.00	0.60	3.00	79.00	0.76
22.00	60.00	0.60	12.10	69.90	0.60	2.00	80.00	0.76
21.00	61.00	0.60	11.90	70.10	0.76	1.00	81.00	0.76
						0.00	82.00	0.76

The metallic pole sustains two working platforms, one situated at 20 m height and the other at the superior extremity. There is still a set of antennas located at 27 m from the base and attached to the body of the pole through metallic devices. The platforms and the supporting devices follow the composition presented in Table 4 where ϕ designate the diameter of the platform. The local assessment revealed the presence of microwave (MW) antennas and of radio frequency (RF), which are listed with the rest of the structure accessories in

Table 5. The data related to the antennas were obtained from the catalogue of the manufacturer. All the aforementioned devices represent additional masses and concentrated forces on the structure, as shown in
Table 6, which presents the structural parameters and the parameters of the existing devices, the specific weight adopted for the material of the structure, the localized and distributed axial load. The geometry of the structure and the existing devices are schematically represented in Figure 3. In Figure 4 they are presented photographic images of the pole.

Table 4. Composition of the platform and support

Platform ϕ = 2.5 m	Mass (kg)
Floor sheet	116
Lateral floor sheet	46
Channel (U) 150 × 12.2 mm – Banister	96
Angle (L) 102 × 76 × 6.4 mm – Banister	68
Angle (L) 102 × 76 × 6.4 mm – Banister	77
Angle (L) 102 × 76 × 6.4 mm – Floor support	43
Platform lower ring	14
Joints	3
Banister bolts	5
Angles (L) 152 × 102 × 9.5 mm – Platform lower support	33
Total =	500
Support set for antenna	**Mass (kg)**
Pipe ϕ = 1´ (25.4 mm)	6
Angle (L) 203 × 152 × 19 mm	50
Staples U (ϕ = 1´ = 25.4 mm)	1
Top plate	1
Total =	58

Table 5. Composition of the localized nodal masses

Device	Mass (kg/unit)	1st Plat (20 m) Quant.	1st Plat (20 m) (kg)	Support (27 m) Quant.	Support (27 m) (kg)	2nd Plat (30 m) Quant.	2nd Plat (30 m) (kg)
Antenna RF 2.6 m	19	2	37	3	56	1	19
Antenna RF 1.23 m	4	1	4	0	0	1	4
Antenna MW	19	2	38	0	0	0	0
Platform	500	1	500	0	0	1	500
Support for antennas	58	6	345	3	173	6	345
Pipe ϕ = 1´ (25.4 mm) (Guide)	6	0	0	0	0	1	6
Pipe ϕ = 3/4´ (19 mm) (LC)	6	0	0	0	0	1	6
Total (kg) =			924		228		880

(LC = Lightning conductor, MW = Microwave, RF = Radio frequency, Plat = Platform)

Table 6. Localized axial load and characteristics of the devices

Device	Frontal area	Height	Weight, distributed weight
Pole	Variable	0-30 m	77 kNm^{-3}
Ladder	0.05 m^2/m	0-30 m	0.15 kN m^{-3}
Cables	0.15 m^2/m	0-30 m	0.25 kN m^{-3}
1st Platform Antenna of the 1st platform	2.60 m^2 1.99 m^2	20 m	9.06 kN
Intermediate antennas Intermediate supports	2.11 m^2 0.56 m^2	27 m	2.24 kN
2nd Platform Antennas of the 2nd platform	2.36 m^2 0.90 m^2	30 m	8.63 kN

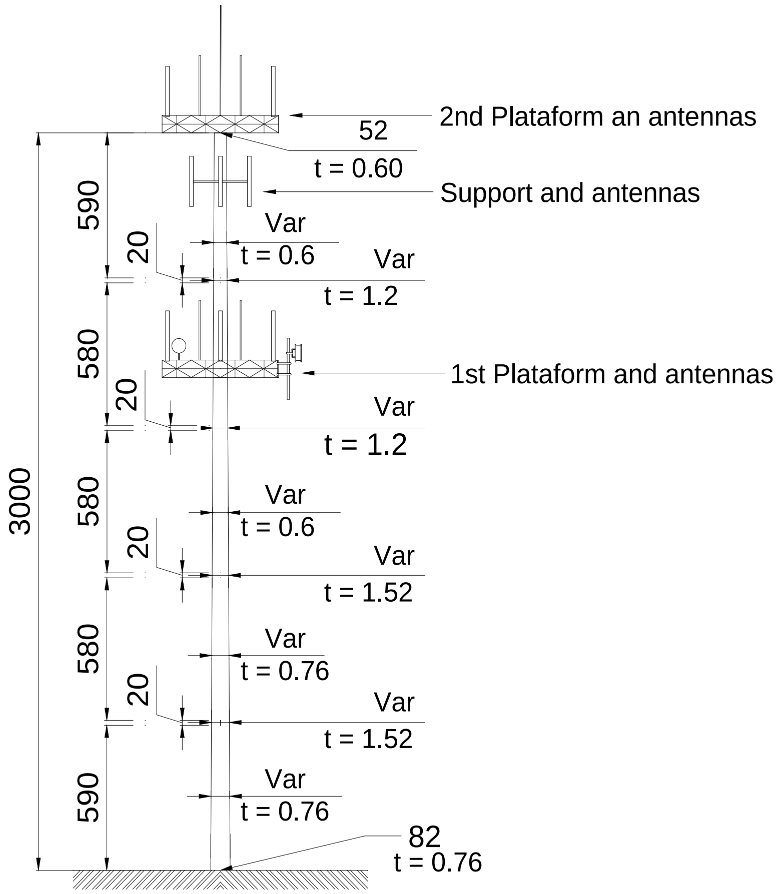


Figure 3. Geometry – Measures in centimeters

Figure 4. General photographic views

The modal shapes obtained by FEM and by the mathematic functions can seem in graph of Figure 5. The exponent of the potential function which best adjusts the curve is 1.85.

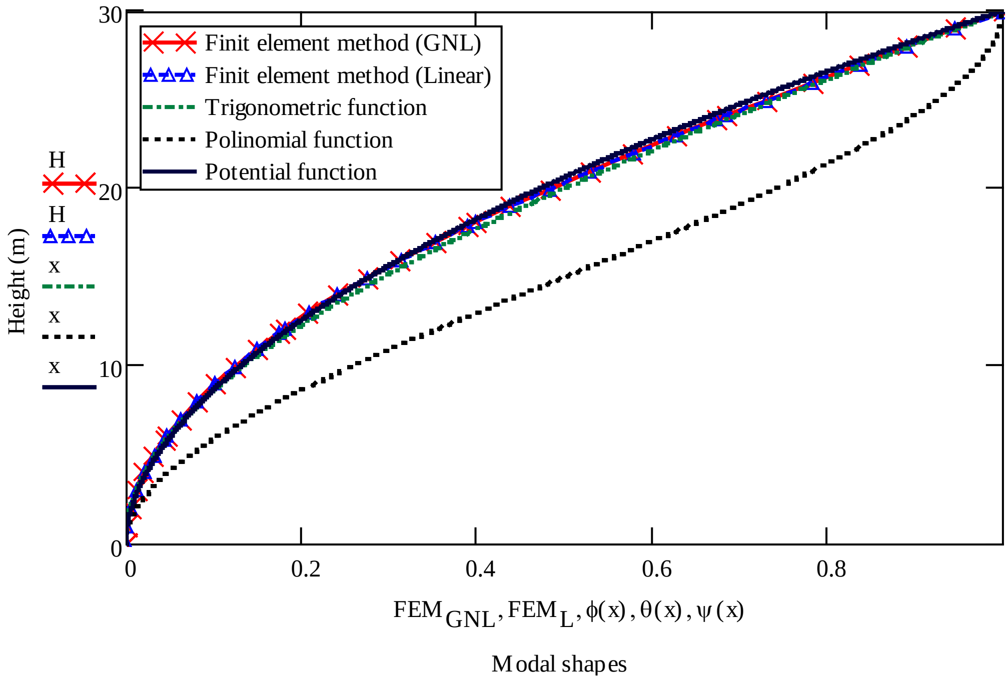


Figure 5. Modal shapes of structure with slenderness 256

Conclusions

In the present study, the shape of the first mode of vibration was investigated using case studies. Analysis by finite element method (FEM) was performed using two different procedures, including a linear procedure, where the geometric stiffness was not considered, and a nonlinear procedure, called the geometric nonlinear formulation (GNL), which considered the geometric stiffness. For comparison, several mathematic functions were studied, and all of the functions were valid throughout the entire domain of the structure.

For the studied cases, geometric stiffness did not have a significant effect on the shape of the first mode of vibration, and the trigonometric function was shown to be a good approximation for the nonlinear vibration shape. The mathematic potential function also represented the first shape of the vibration. For the structure with a slenderness index of 310, the exponent of the function was equal to 1.965, while the structure with a slenderness index of 256 corresponded to an exponent of 1.865. With this information, the weight-averaged rate of slenderness (r_s) was determined to be $r_s = 0.006812$. Thus, an adequate exponent could be obtained by multiplying the slenderness index by r_s. For example, for a structure with a slenderness of 200, the exponent is equal to 1.36 (200 times 0.006812).

Finally, the polynomial function did not provide an accurate representation of the vibration shape of the first mode.

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