Averaging Navier-Stokes equations system by a dual approach

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Abstract

The current three-dimensional averaging mathematical model of flow, also known as the Reynolds equations, was developed based on the idea of Reynolds in 1895. This model is given by the classical averaging of velocity and pressure parameters from the three-dimensional Navier-Stokes equations system. However, by doing this, these averaging parameters obtained by this classical approach are not generalized in comparison to ones estimated by the dual approach. This paper proposes a dual approach to establishing the three-dimensional flow equation. The model setup is more complicated than the classical model in terms of integration because the procedure can be repeated several times. In this paper, the authors perform twice: (i) first, integration of the velocity and pressure parameters from time t to Tm, with time Tm < t+T, where T is the repeated period of parameters; (ii) second, integration from time t to t+T. Fluctuating quantities such as velocity and pressure in turbulent flow, over time, are simulated using trigonometric Fourier series. The three-dimensional flow model obtained from this dual approach could provide more accurate results than those given by the Reynolds equations.

Keywords: Classical averaged method, dual approach, three-dimensional Navier-Stokes equations

Introduction

Flow in nature is usually turbulent and three-dimensional (3D) and it is often described by the Reynolds equations system [1, 2], classically averaged from the 3D Navier-Stokes equations system [1, 2].

With the classical averaging, quantities such as velocity and pressure are simply arithmetically averaged. In this research, these quantities will be averaged based on the dual approach, meaning that physical quantities such as velocity, pressure are integrated many times, have both local and global integration [3, 4]. In this paper, they are integrated only twice.

With this dual approach, it will be more complicated than the classical approach, but in return, we will receive better physical flow quantities than the classical approach [3, 4, 5, 6, 7, 8].

Averaging Navier-Stokes equations by a dual approach

The 3D Navier-Stokes equations [2] describes liquid motion written as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \mathrm{w}}{\partial z}=0 \tag{1a}$$

$$\frac{\partial u}{\partial t}+\frac{\partial (u.u)}{\partial x}+\frac{\partial (u.v)}{\partial y}+\frac{\partial (u.\mathrm{w})}{\partial z}=\frac{1}{\rho}.F_x-\frac{1}{\rho}.\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} \tag{1b}$$

$$\frac{\partial v}{\partial t}+\frac{\partial (u.v)}{\partial x}+\frac{\partial (v.v)}{\partial y}+\frac{\partial (v.\mathrm{w})}{\partial z}=\frac{1}{\rho}.F_y-\frac{1}{\rho}.\frac{\partial p}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{yy}}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z} \quad (1c)$$

$$\frac{\partial \mathrm{w}}{\partial t}+\frac{\partial (u.\mathrm{w})}{\partial x}+\frac{\partial (v.\mathrm{w})}{\partial y}+\frac{\partial (\mathrm{w.w})}{\partial z}=\frac{1}{\rho}.F_z-\frac{1}{\rho}.\frac{\partial p}{\partial z}+\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial y}+\frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z} \quad (1d)$$

With turbulent flow, when the velocity and pressure quantities are averaged by the classical method, we get the Reynolds equations system as follows [1, 2]:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{\mathrm{w}}}{\partial z}=0 \quad (2a)$$

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial (\bar{u}.\bar{u})}{\partial x}+\frac{\partial (\bar{u}.\bar{v})}{\partial y}+\frac{\partial (\bar{u}.\bar{\mathrm{w}})}{\partial z}=\frac{1}{\rho}.F_x-\frac{1}{\rho}.\frac{\partial \bar{p}}{\partial x}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{xx}}{\partial x}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{xy}}{\partial y}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{xz}}{\partial z}-\frac{\partial (\overline{u'.u'})}{\partial x}-\frac{\partial (\overline{u'.v'})}{\partial y}-\frac{\partial (\overline{u'.\mathrm{w}'})}{\partial z} \quad (2b)$$

$$\frac{\partial \bar{v}}{\partial t}+\frac{\partial (\bar{v}.\bar{u})}{\partial x}+\frac{\partial (\bar{v}.\bar{v})}{\partial y}+\frac{\partial (\bar{v}.\bar{\mathrm{w}})}{\partial z}=\frac{1}{\rho}.F_y-\frac{1}{\rho}.\frac{\partial \bar{p}}{\partial y}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{yx}}{\partial x}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{yy}}{\partial y}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{yz}}{\partial z}-\frac{\partial (\overline{v'.u'})}{\partial x}-\frac{\partial (\overline{v'.v'})}{\partial y}-\frac{\partial (\overline{v'.\mathrm{w}'})}{\partial z} \quad (2c)$$

$$\frac{\partial \bar{\mathrm{w}}}{\partial t}+\frac{\partial (\bar{\mathrm{w}}.\bar{u})}{\partial x}+\frac{\partial (\bar{\mathrm{w}}.\bar{v})}{\partial y}+\frac{\partial (\bar{\mathrm{w}}.\bar{\mathrm{w}})}{\partial z}=\frac{1}{\rho}.F_z-\frac{1}{\rho}.\frac{\partial \bar{p}}{\partial z}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{zx}}{\partial x}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{zy}}{\partial y}+\frac{1}{\rho}\frac{\partial \bar{\tau}_{zz}}{\partial z}-\frac{\partial (\overline{\mathrm{w}'.u'})}{\partial x}-\frac{\partial (\overline{\mathrm{w}'.v'})}{\partial y}-\frac{\partial (\overline{\mathrm{w}'.\mathrm{w}'})}{\partial z} \quad (2d)$$

Where:

$u=\bar{u}+u'$, $v=\bar{v}+v'$, $\mathrm{w}=\bar{\mathrm{w}}+\mathrm{w}'$, $p=\bar{p}+p'$

u, v, w, p are the instantaneous velocity (in the x, y and z direction respectively) and p is the pressure;

$\bar{u}, \bar{v}, \bar{\mathrm{w}}, \bar{p}$ are the averaged quantities of velocity (in the x, y and z direction respectively) and p is pressure;

u', v', w', p' are the fluctuating quantities of velocity (in the x, y, and z - direction respectively) and pressure;

$\overline{y_i'.y_j'}$ is the mean product of fluctuating quantities of y_i, y_j; is the friction components generated by the turbulent flow; y_i, y_j represent velocity u, v, w or pressure p which can be approximated in a variety of ways [2];

$\bar{\tau}_{i,j}$ is the shear stress, where the subscript i, j represent the x, y or z directions; if $i \equiv j$ we have the normal stress; if $i \neq j$ we have the shear stress.

In this paper, the average pressure and velocity quantities are obtained by dual approach by twice integration [3, 4] as follows:

$$\bar{y}(t)=\frac{1}{T}\int_0^T\frac{1}{r}\int_0^r y(t).dt.dr \quad (3)$$

First integral: these quantities are averaged by integration from time t (for simplicity we choose t = 0) to time (t + r), where r < T, and T is the fluctuating period of the quantities to be integrated:

$$\bar{y}_r=\frac{1}{r}\int_0^r y(t).dt \quad (4)$$

Second integral: integrations of these quantities are performed from time t to t+T:

$$\bar{y}(t)=\frac{1}{T}\int_0^T \bar{y}_r.dr \quad (5)$$

The fluctuating pressure or velocity quantities are approximated as a trigonometric Fourier series [9, 10, 11]; so the instantaneous velocity or pressure y(t) is approximate as:

$$y(t) = A_0/2 + \sum_{p=1}^{\infty}[\mathrm{A}_p.\cos(\mathrm{p}\omega \mathrm{t}) + B_p.\sin(\mathrm{p}\omega \mathrm{t}) \tag{6}$$

where:

$$A_0 = \frac{2}{T}\int_0^T y(t)dt;$$

$$A_p = \frac{2}{T}\int_0^T y(t).\cos(\mathrm{p}\omega \mathrm{t});\ p = 1, 2, ...$$

$$B_p = \frac{2}{T}.\int_0^T y(t).\sin(\mathrm{p}\omega \mathrm{t})dt;\ p = 1, 2, ...$$

Substituting the quantity $y(t)$ into (4), we have the average quantity $\overline{y}_r(t)$ as follows:

$$\overline{y}_r(t) = A_0/2 + \sum_{p=1}^{\infty}[\frac{\mathrm{A}_p}{p\omega T_m}.\sin(\mathrm{p}\omega \mathrm{T}_m) - \frac{\mathrm{B}_p}{p\omega T_m}.\cos(\mathrm{p}\omega \mathrm{T}_m)]$$

and then, substituting $\overline{y}_r(t)$ into (5), we receive the average value $\overline{y}_j = \overline{y}(t_j)$ by dual approach as follows:

$$\overline{y}_j \equiv \overline{y}(t_j) = A_0/2 - \frac{1}{2\Pi T^2}\left(\sum_{p=1}^{N/2}\frac{B_p}{p}\right)$$

In this paper, the instantaneous velocity or pressure quantities at time t = t_n , denoted by $y(t_n)$ are discrete and finite measurement data; so we have the corresponding variables:

$$\frac{t}{T} \equiv \frac{t_n}{T} \sim \frac{n}{N} \tag{7}$$

Therefore, they are approximated by a discrete, finite trigonometric Fourier series as follows [10, 11]:

$$y(t_n) = A_0/2 + \sum_{p=1}^{N/2}[\mathrm{A}_p.\cos(2\pi \mathrm{pn/N}) + B_p.\sin(2\pi \mathrm{pn/N}) \quad ; \quad n = 1, 2, ..., N \tag{8}$$

where:

A_0, A_p, B_p are the coefficients;

$$A_0 = \frac{2}{N}\sum_{n=1}^{N} y_n; \qquad B_0 = 0 \tag{9a}$$

$$A_p = \frac{2}{N}\sum_{n=1}^{N}[\mathrm{y}_n.\cos(2\pi \mathrm{pn/N});\ p = 1, 2, ...N/2 \tag{9b}$$

$$B_p = \frac{2}{N}\sum_{n=1}^{N}[\mathrm{y}_n.\sin(2\pi \mathrm{pn/N});\ p = 1, 2, ...N/2 \tag{9c}$$

N sum of instantaneous velocity or pressure data values (u, v, w, p) mesures in time T;

p indicates the pth (for p > N/2 these trigonometric harmonics will repeat);

t_n time to calculate, $t_n = n.\Delta t$;

n time increment step.

The average velocity or pressure value obtained by the dual approach $\overline{y}_j = \overline{y}(t_j)$ will be as follows:

$$\overline{y}_j \equiv \overline{y}(t_j) = A_0/2 + \Delta DA \tag{10}$$

Where:

$A_0/2$ is the average quantity of Σy_j, calculated by the original ideal of Reynolds (1895), also called classical RANS equations;

$$\Delta DA = \frac{-1}{2\Pi N^2}\left(\sum_{p=1}^{N/2}\frac{B_p}{p}\right) \tag{11}$$

ΔDA is the difference of velocity or pressure quantities, when averaging these quantities with respect to the dual approach, compared with the classical RANS equations.

Algorithm used to solve the 3D Navier-Stokes equations system which is averaged by the dual approach

There are two methods to solve the 3D Naviers-Stokes equations system which is averaged by the dual approach.

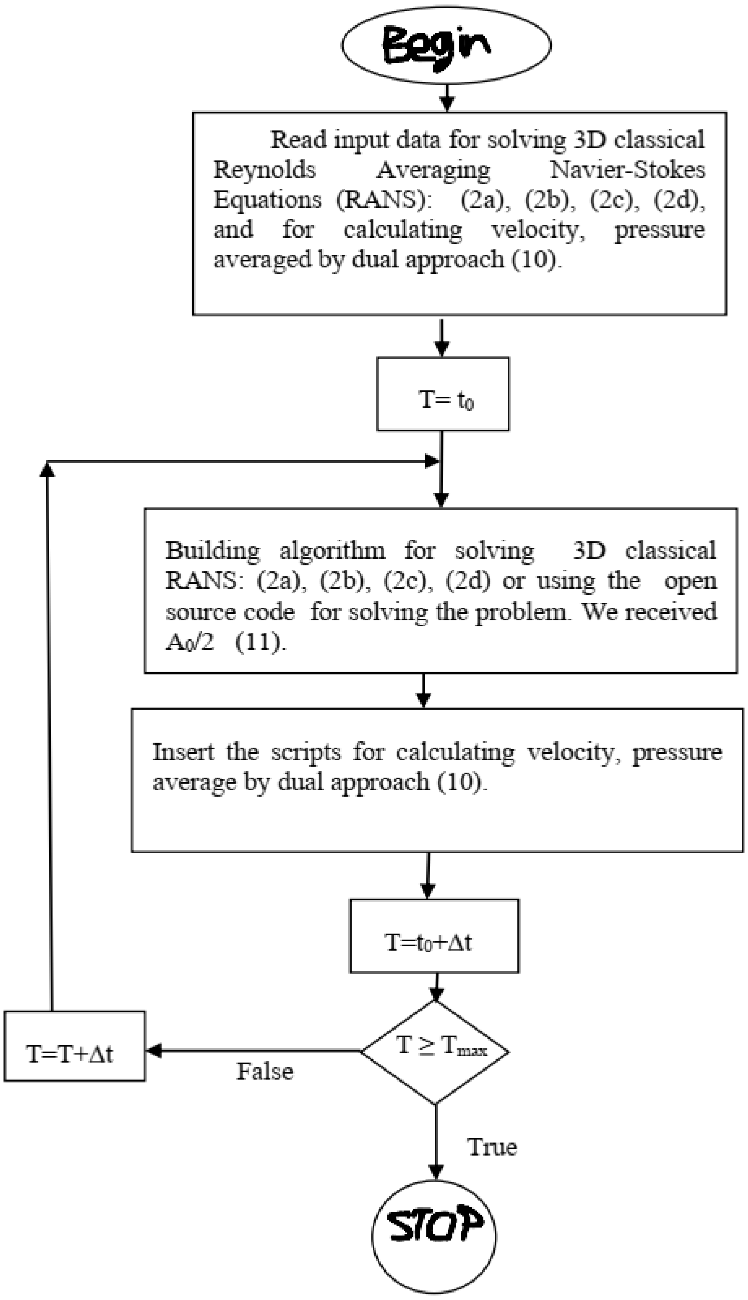

Figure 1. Flow chart of solving the 3D averaging Naviers-Stokes equations by dual approach

(i) *Method 1*: By relying on the numerical solution results of the 3D RANS equations by open source code, inserting command line segments to add differential increments ΔDA(11) to the velocity and pressure solutions calculated at each time step to obtain the numerical solution of the 3D Naviers-Stokes equations system averaging by the dual approach, formula (10) (see Step 3, Fig. 1).

(ii) *Method 2*: Completely build a new algorithm to solve the 3D Naviers-Stokes equations system which is averaged by the dual approach as follows:

At each time step, constructing an algorithm to solve the 3D classical RANS equations, after that, inserting command line segments to add differential increments ΔDA (11) to the velocity and pressure solutions calculated at each time step to obtain the numerical solution of the 3D Naviers-Stokes equations averaging by the dual approach, formula (10) (see Step 3, Fig. 1).

Case study

In this case study, we approximate the instantaneous pressure quantity by a finite trigonometric Fourier series (8):

$$y(t_n) = A_0/2 + \sum_{p=1}^{N/2} [\mathrm{A}_p.\cos(2\pi \mathrm{pn/N}) + B_p.\sin(2\pi \mathrm{pn/N}); \quad n = 1, 2, ..., N \tag{8}$$

The instantaneous pressure at a fixed point (MC7-2) measured over time is given in Table 1 and is shown in Fig. 2.

Table 1: Pressure measurement data over time at a fixed point (MC7-2) in the first twenty (N = 20) measurement steps [12]

Time (sec)	1/5	2/5	3/5	4/5	5/5
Pressure (mm)	354.5706523	356.9267077	359.4528209	357.827718	354.8137758
Time (sec)	6/5	7/5	8/5	9/5	10/5
Pressure (mm)	354.4195018	357.937984	359.5774499	360.402683	360.8192235
Time (sec)	11/5	12/5	13/5	14/5	15/5
Pressure (mm)	359.196406	358.0766566	358.3154676	356.2717004	356.2604194
Time (sec)	16/5	17/5	18/5	19/5	20/5
Pressure (mm)	359.5143341	361.8665435	360.5558524	359.1800103	357.4757616

With the pressure measurement interval over time ΔT=1/5 sec, the arithmetic mean pressure value (classical RANS) is:

$$\overline{p}_{CA} = A_0/2 = \frac{1}{N}\sum_{p_i=1}^{N} p_i = 358.1730834mm;$$

Average value of pressure according to dual approach is:

$$\overline{y}_j \equiv \overline{y}(t_j) = \overline{p_{CA}} + \Delta DA = \overline{p_{CA}} - \frac{1}{2\Pi N^2}\left(\sum_{p=1}^{N/2}\frac{B_p}{p}\right) = A_0/2 - \frac{1}{2\Pi N^2}\left(\sum_{p=1}^{N/2}\frac{B_p}{p}\right) \tag{10}$$

$$\overline{y}_j = 358.1730834 - 0.4384605 = 357.7346229mm$$

Comment

(i) From the solution obtained by the dual approach (10), when we arithmetically average the fluctuating velocity and pressure quantities as Reynolds proposed in 1895 (i.e. for $\Delta DA = 0$), we get the classical RANS equations [1, 2].

(ii) In order to calculate the increment ΔDA, we need to known the values of the fluctuating velocity and pressure quantities in the cycle T.

(iii) The results here are for the case of a normal measuring point in the flow; in fact there are cases where the flow has a very large fluctuating values; and so the increments ΔDA will also be considerably large.

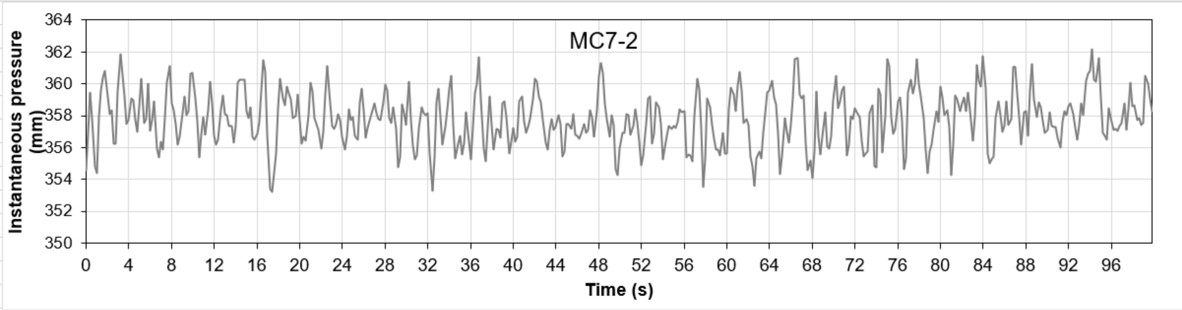

Figure 2. Pressure fluctuation at measuring point MC7-2 over time [12]

Conclusion

In this paper, the velocity and pressure fluctuations are approximated by a trigonometric Fourier series; the velocity or pressure quantities received by the dual approach (10) are more general than those obtained using the classical RANS method.

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