A dual-reciprocity boundary element method for axisymmetric thermoelastodynamic deformations in functionally graded solids

***W. T. Ang and B. I. Yun**

Division of Engineering Mechanics, School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore.

*Corresponding author: wtang@pmail.ntu.edu.sg

Abstract

A dual-reciprocity boundary element method is outlined for solving a class of initial-boundary value problems concerning axisymmetric thermoelastodynamic deformations in functionally graded materials. The time derivatives of the temperature and the displacement, which appear in the governing partial differential equations, are suppressed by using the Laplace transformation technique. In the Laplace transform domain, the problem under consideration is formulated in terms of integral equations which contain both boundary integrals and domain integrals. The dual-reciprocity method is used together with suitably constructed interpolating functions to reduce the domain integrals approximately into boundary integrals. The problem under consideration is eventually reduced to linear algebraic equations which may be solved for the numerical values of the Laplace transforms of the temperature and the displacements at selected points in space. The temperature and the displacement in the physical time domain are approximately recovered by using a numerical method for inverting Laplace transforms. To check that the numerical procedure presented is valid, it is applied to solve a specific test problem which has a closed-form analytic solution.

Keywords: Boundary element method, Dual-reciprocity method, Interpolating functions, Laplace transformation, Axisymmetric thermoelasticity, Functionally graded materials.

Introduction

In recent years, there has been considerable interest in the analysis of axisymmetric materials possessing material properties that are graded continuously along the axial and radial directions. For example, Clements and Kusuma (2011) studied the axisymmetric deformation of an elastic half space having elastic moduli that vary as a quadratic function of the axial coordinate; Matysiak, Kulchytsky-Zhyhailo and Perkowski (2011) considered the Reissner-Sagoci problem for a homogeneous layer bonded to an elastic half space with a shear modulus that varies axially in accordance with a simple power law; and Keles and Tutuncu (2011) calculated the dynamic displacement and stress fields in hollow cylinders and spheres with material properties that are functionally graded along the axial direction by a simple power law.

In the present paper, the dual-reciprocity boundary element approach and the interpolating functions proposed in Yun and Ang (2012) for solving an axisymmetric thermoelastostatic problem involving functionally graded materials is extended to thermoelastodynamic deformations. The material properties vary with the axial and radial coordinates following sufficiently smooth functions in general forms.

It may be of interest to note that a boundary element solution of the corresponding two-dimensional thermoelastodynamic problem for functionally graded solids may be found in a very recent paper by Ekhlakov. Khay, Zhang, Sladek and Sladek (2012).

Basic equations of axisymmetric thermoelastodynamics

With reference to the cylindrical polar coordinates r, θ and z, the temperature T and the displacement $\underline{\mathbf{u}}$ in an isotropic solid that is symmetrical about the z axis is independent of θ and the only non-zero components of the displacement $\underline{\mathbf{u}}$ are given by u_r and u_z. If the material properties of the solid are radially and axially graded using sufficiently smooth functions of r and z, the governing partial differential equations of axisymmetric thermoelastodynamics are given by

$$\underline{\nabla}\cdot(\kappa\underline{\nabla}T)+Q=\beta T_0\frac{\partial}{\partial t}[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}]+\rho c\frac{\partial T}{\partial t}, \tag{1}$$

$$\begin{aligned}&\nabla^2_{\text{axis}}u_r-\frac{u_r}{r^2}+\frac{1}{1-2\nu}\frac{\partial}{\partial r}(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z})\\&=\frac{1}{\mu}\{\beta\frac{\partial T}{\partial r}+\frac{\partial\beta}{\partial r}(T-T_0)-F_r-\frac{\partial\mu}{\partial z}(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r})\\&-2\frac{\partial\mu}{\partial r}[\frac{\partial u_r}{\partial r}+\frac{\nu}{1-2\nu}(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z})]+\rho\frac{\partial^2 u_r}{\partial t^2}\},\end{aligned} \tag{2}$$

$$\begin{aligned}&\nabla^2_{\text{axis}}u_z+\frac{1}{1-2\nu}\frac{\partial}{\partial z}(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z})\\&=\frac{1}{\mu}\{\beta\frac{\partial T}{\partial z}+\frac{\partial\beta}{\partial z}(T-T_0)-F_z-\frac{\partial\mu}{\partial r}(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r})\\&-2\frac{\partial\mu}{\partial z}[\frac{\partial u_z}{\partial z}+\frac{\nu}{1-2\nu}(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z})]+\rho\frac{\partial^2 u_z}{\partial t^2}\},\end{aligned} \tag{3}$$

where $\nabla^2_{\text{axis}}\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$, t is the time coordinate, T_0 is a constant reference temperature at which the body does not experience any thermally induced stress, the coefficients κ, β, ρ, c, ν and μ are respectively the thermal conductivity, stress-temperature coefficient, density, specific heat capacity, Poisson's ratio and shear modulus of the isotropic body, F_r and F_z are respectively the r and the z components of the body force, and Q is the internal heat generation term. Note that κ, β, ρ, c and μ are, in general, functions of r and z and the Poisson's ratio ν is assumed to be constant. The body force components F_r and F_z and the internal heat generator Q are, in general, functions of the axisymmetric coordinates r and z and the time coordinate t.

Details on the basic equations of thermoelasticity may be found in Nowacki (1986).

Boundary-domain integral equations

The governing partial differential equations in (1), (2) and (3) in terms of the boundary-domain integral equations

$$
\begin{aligned}
&\gamma(r_0,z_0)\sqrt{\kappa(r_0,z_0)}T(r_0,z_0,t)\\
&=\int_\Gamma\{T(r,z,t)[\sqrt{\kappa(r,z)}G_1(r,z;r_0,z_0;n_r,n_z)-\frac{\partial\sqrt{\kappa(r,z)}}{\partial n}G_0(r,z;r_0,z_0)]\\
&-\sqrt{\kappa(r,z)}G_0(r,z;r_0,z_0)q(r,z,t;n_r,n_z)\}rds(r,z)\\
&+\int\int_\Omega G_0(r,z;r_0,z_0)[-\frac{Q(r,z,t)}{\sqrt{\kappa(r,z)}}+T(r,z,t)\cdot\nabla^2_{\text{axis}}(\sqrt{\kappa(r,z)})\\
&+\frac{\rho(r,z)c(r,z)}{\sqrt{\kappa(r,z)}}\frac{\partial T(r,z,t)}{\partial t}\\
&+\frac{\beta(r,z)T_0}{\sqrt{\kappa(r,z)}}\frac{\partial}{\partial t}[\frac{\partial u_r(r,z,t)}{\partial r}+\frac{u_r(r,z,t)}{r}+\frac{\partial u_z(r,z,t)}{\partial z}]rdrdz\\
&\qquad\text{for }(r_0,z_0)\in\Omega\cup\Gamma,
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&\gamma(r_0,z_0)u_K(r_0,z_0,t)\\
&=\int_\Gamma(\Phi_{JK}(r,z;r_0,z_0)p_J(r,z,t;n_r,n_z)\\
&-\Psi_{JK}(r,z;r_0,z_0;n_r,n_z)u_J(r,z,t))rds(r,z)\\
&+\int\int_\Omega\frac{1}{\mu(r,z)}\Phi_{JK}(r,z;r_0,z_0)\{-\beta(r,z)\frac{\partial}{\partial x_J}[T(r,z,t)]\\
&\quad-\frac{\partial}{\partial x_J}[\beta(r,z)](T(r,z,t)-T_0)+F_J(r,z,t)\\
&+\frac{\partial}{\partial x_J}[\mu(r,z)]\frac{2\nu}{(1-2\nu)r}[u_r(r,z,t)]\\
&\quad+X_{JN}(r,z)\frac{\partial}{\partial z}[u_N(r,z,t)]+Y_{JN}(r,z)\frac{\partial}{\partial r}[u_N(r,z,t)]\\
&-\rho(r,z)\frac{\partial^2u_K(r,z,t)}{\partial t^2}\}rdrdz\\
&\qquad\text{for }(r_0,z_0)\in\Omega\cup\Gamma\quad(K=r,z),
\end{aligned}
\tag{5}
$$

where Ω is the solution domain on the Orz plane, Γ is the boundary of Ω (excluding the part that lies on the z axis), n_r and n_z are respectively the r and z components of the unit normal outward vector to curve Γ at the point (r,z), $G_0(r,z;r_0,z_0)$ is the fundamental solution of axisymmetric Laplace's equation, $G_1(r,z;r_0,z_0;n_r,n_z)$ is the normal derivative of $G_0(r,z;r_0,z_0)$ along the direction of the vector $[n_r,n_z]$, the uppercase Latin subscripts (such as K) are assigned values r and z and summation over those values are implied for repeated subscripts, $\Phi_{JK}(r,z;r_0,z_0)$ is the fundamental solution of the partial differential equations for axisymmetric elastostatics, $\Psi_{JK}(r,z;r_0,z_0;n_r,n_z)$ is the traction function corresponding to $\Phi_{JK}(r,z;r_0,z_0)$, and $p_J(r,z,t;n_r,n_z)$, $X_{JN}(r,z)$ and $Y_{JN}(r,z)$ are defined by

$$\begin{aligned}
p_r(r,z,t;n_r,n_z) &= 2(\frac{\partial u_r}{\partial r}+\frac{\nu}{1-2\nu}[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}])n_r(r,z) \\
&\quad +(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r})n_z(r,z), \\
p_z(r,z,t;n_r,n_z) &= (\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r})n_r(r,z) \\
&\quad +2(\frac{\partial u_z}{\partial z}+\frac{\nu}{1-2\nu}[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}])n_z(r,z),
\end{aligned} \tag{6}$$

$$\begin{aligned}
X_{rr}(r,z) &= \frac{\partial\mu(r,z)}{\partial z},\ X_{rz}(r,z)=\frac{2\nu}{1-2\nu}\frac{\partial\mu(r,z)}{\partial r}, \\
X_{zr}(r,z) &= \frac{\partial\mu(r,z)}{\partial r},\ X_{zz}(r,z)=\frac{\partial\mu(r,z)}{\partial z}\frac{2(1-\nu)}{1-2\nu}, \\
Y_{rr}(r,z) &= \frac{\partial\mu(r,z)}{\partial r}\frac{2(1-\nu)}{1-2\nu},\ Y_{rz}(r,z)=\frac{\partial\mu(r,z)}{\partial z}, \\
Y_{zr}(r,z) &= \frac{2\nu}{1-2\nu}\frac{\partial\mu(r,z)}{\partial z},\ Y_{zz}(r,z)=\frac{\partial\mu(r,z)}{\partial r}.
\end{aligned} \tag{7}$$

The functions $p_J(r,z,t;n_r,n_z)$ are related to the axisymmetric tractions $t_J(r,z,t;n_r,n_z)$ through

$$\begin{aligned}
t_J(r,z,t;n_r,n_z) &= \mu(r,z)p_J(r,z,t;n_r,n_z) \\
&\quad -\beta(r,z)[T(r,z,t)-T_0]\delta_{JL}n_L(r,z),
\end{aligned} \tag{8}$$

where δ_{JN} is the Kronecker-delta.

The boundary-domain integral equations in (4) and (5) for the corresponding case of axisymmetric thermoelastostatic deformations are given in Yun and Ang (2012) where the details of the functions $G_0(r,z;r_0,z_0)$, $G_1(r,z;r_0,z_0;n_r,n_z)$, $\Phi_{JK}(r,z;r_0,z_0)$ and $\Psi_{JK}(r,z;r_0,z_0;n_r,n_z)$ are explicitly written out.

Dual-reciprocity boundary element method

The dual-reciprocity method in Partridge, Brebbia and Wrobel (1992) may be employed to approximate the domain integrals over Ω in the integral equations (4) and (5) in terms of boundary integrals over the curve Γ by using interpolating functions centered about selected collocation points in $\Omega \cup \Gamma$. As in Yun and Ang (2012), the collocating functions centered about the n-th collocation point, denoted by $\chi^{(n)}(r,z)$, $\phi^{(n)}(r,z)$, $\chi_{KJ}^{(n)}(r,z)$ and $\phi_{KJ}^{(n)}(r,z)$, are assumed to be sufficiently smooth and are required to satisfy the partial differential equations

$$
\begin{aligned}
&\nabla_{\text{axis}}^2 \chi^{(n)}(r,z) = \phi^{(n)}(r,z), \\
&\nabla_{\text{axis}}^2 \chi_{rJ}^{(n)}(r,z) - \frac{\chi_{rJ}^{(n)}(r,z)}{r^2} \\
&+ \frac{1}{1-2\nu}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}[\chi_{rJ}^{(n)}(r,z)] + \frac{\chi_{rJ}^{(n)}(r,z)}{r} + \frac{\partial}{\partial z}[\chi_{zJ}^{(n)}(r,z)]\right) = \phi_{rJ}^{(n)}(r,z), \\
&\nabla_{\text{axis}}^2 \chi_{zJ}^{(n)}(r,z) \\
&+ \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial r}[\chi_{rJ}^{(n)}(r,z)] + \frac{\chi_{rJ}^{(n)}(r,z)}{r} + \frac{\partial}{\partial z}[\chi_{zJ}^{(n)}(r,z)]\right) = \phi_{zJ}^{(n)}(r,z).
\end{aligned}
$$

(9)

In Agnantiaris, Polyzos and Beskos (2001) and Wang, Mattheij and ter Morsche (2003), the interpolating functions $\chi^{(n)}(r,z)$, $\phi^{(n)}(r,z)$, $\chi_{KJ}^{(n)}(r,z)$ and $\phi_{KJ}^{(n)}(r,z)$ are constructed by integrating axially selected radial basis functions in three-dimensional space. The interpolating functions thus constructed are well defined at $r=0$, but they are in highly complicated forms and are expressed in terms of special functions given by the elliptic integrals.

To construct interpolating functions expressed in terms of relatively simple elementary functions, one may choose $\chi^{(n)}(r,z)$ and $\chi_{KJ}^{(n)}(r,z)$ to be sufficiently smooth functions of $\sqrt{(r-r_0^{(n)})^2+(z-z_0^{(n)})^2}$, where $(r_0^{(n)}, z_0^{(n)})$ is the n-th collocation point, and determine $\phi^{(n)}(r,z)$ and $\phi_{KJ}^{(n)}(r,z)$ using (9). Nevertheless, the interpolating functions $\phi^{(n)}(r,z)$ and $\phi_{KJ}^{(n)}(r,z)$ constructed in this manner are not well defined at $r=0$. This poses a problem if the z axis is part of the solution domain Ω. In Yun and Ang (2012), the singular behaviors of $\phi^{(n)}(r,z)$ and $\phi_{KJ}^{(n)}(r,z)$ at $r=0$ are removed by modifying $\chi^{(n)}(r,z)$ and $\chi_{KJ}^{(n)}(r,z)$ in such a way that $\chi^{(n)}(r,z)$ and $\chi_{KJ}^{(n)}(r,z)$ behave as $O(r^2)$ for small r. Specifically, $\chi^{(n)}(r,z)$ and $\chi_{KJ}^{(n)}(r,z)$ are taken to be

$$
\begin{aligned}
&\chi^{(n)}(r,z) = \frac{1}{9}\{[\sigma(r,z;r_0^{(n)},z_0^{(n)})]^3 + [\sigma(r,z;-r_0^{(n)},z_0^{(n)})]^3\}, \\
&\chi_{rr}^{(n)}(r,z) = \chi^{(n)}(r,z) - \frac{2}{9}[\sigma(0,z;r_0^{(n)},z_0^{(n)})]^3, \\
&\chi_{zr}^{(n)}(r,z) = \chi_{rz}^{(n)}(r,z) = 0,\ \chi_{zz}^{(n)}(r,z) = \chi^{(n)}(r,z),
\end{aligned}
\tag{10}
$$

where $\sigma(r,z,r_0,z_0) = \sqrt{(r-r_0)^2+(z-z_0)^2}$.

For a numerical procedure for solving initial-boundary value problems governed by (1), (2) and (3), we apply the Laplace transformation on the boundary-domain integral equations (4) and (5) to suppress the time derivatives of T, u_r and u_z, use the dual-reciprocity method together with the interpolating functions constructed using (10) to approximate the domain integrals in the resulting boundary-domain integral equations in terms of boundary integrals, and discretize the boundary Γ into elements to develop a boundary element procedure for finding the temperature and the displacement in the Laplace transform domain. The temperature and the displacement in the physical domain may be recovered by using a numerical method for inverting Laplace transforms.

Test problem

The coefficients of the partial differential equations in (1), (2) and (3) are chosen to be given by $\rho = r^2 + z^2$, $c = 2$, $\beta = r^2 + z^2$, $\mu = r + z$, $\kappa = (r^2 + z^2)^2$, $\nu = 3/10$, and

$$\begin{aligned}
Q(r,z,t) &= -\frac{r^2+z^2}{r}\{\sin(t)[16r^3 z - r^2 + 4r + 4rz^3 + z^2] \\
&\quad -2r^3 z\cos(t) + 16r^3 z + 4rz^3\}, \\
F_r(r,z,t) &= -\frac{1}{2r^2}\{\cos(t)[4r^5 + 2r^4 z^2 + 4r^3 z^2 - 4r^3 \\
&\quad +2r^2 z^2 + 20r^2 + r^2 z - 4rz^2 - 7z^3] \\
&\quad +(1+\sin(t))[-8r^5 z - 4r^3 z^3] + 4r^3\}, \\
F_z(r,z,t) &= \frac{1}{2r^2}\{\cos(t)[2r^4 z - 4r^3 + 7r^2 + 2r^2 z^3 \\
&\quad -10rz - 4rz^2 - 12r - 11z^2] \\
&\quad +(1+\sin(t))[2r^5 + 6r^3 z^2] - 4rz\}.
\end{aligned}$$

It is easy to check that a solution of the partial differential equations is given by

$$\begin{aligned}
T(r,z,t) &= r^2 z(\sin(t)+1), \\
u_r(r,z,t) &= (z^2+2r)\cos(t), \\
u_z(r,z,t) &= (2-rz)\cos(t).
\end{aligned} \tag{11}$$

For a specific initial-boundary value problem as a test problem, take the solution domain Ω to be $1 < r < 2,\ 0 < z < 1,$ which is a rectangular region on the Orz plane, and use the solution in (11) to generate the following initial and boundary data – (a) initial values of T, u_r, u_z, $\frac{\partial u_r}{\partial t}$ and $\frac{\partial u_z}{\partial t}$ at time $t = 0$ at points (r,z) in $\Omega \cup \Gamma$, (b) boundary values of the displacement (u_r, u_z) on the entire boundary of Ω for time $t > 0$, (c) boundary values of T on the sides of the rectangular region where $z = 0$ and $z = 1$ for time $t > 0$, and (d) boundary values of $\frac{\partial T}{\partial r}$ on the sides of the rectangular region where $r = 1$ and $r = 2$ for time $t > 0$.

For the boundary element procedure, the sides of the rectangular region are discretized into 80 straight line elements. The Laplace transforms of the temperature, heat flux, displacement and traction on the boundary elements are approximated using discontinuous linear functions. As many as 121 well distributed collocation points in $\Omega \cup \Gamma$ (including those on the boundary elements) are used in the dual-reciprocity method for converting approximately the domain integrals in the integral formulation of the initial-boundary value problem into boundary integrals. We use the numerical method in Stehfest (1970) to invert the Laplace transforms in order to recover the temperature and the displacement in the physical domain.

Numerical values of T, u_r and u_z obtained using the dual-reciprocity boundary element method (DRBEM) are plotted against t ($0 < t < 6$) at $(r, z) = (1.5, 0.5)$ in Figures 1, 2 and 3 respectively. The numerical values agree well with the analytical solution in (11), showing that the interpolating functions given in (9) and (10) are employed successfully to treat the domain integrals in the boundary-domain integral equations in (4) and (5).

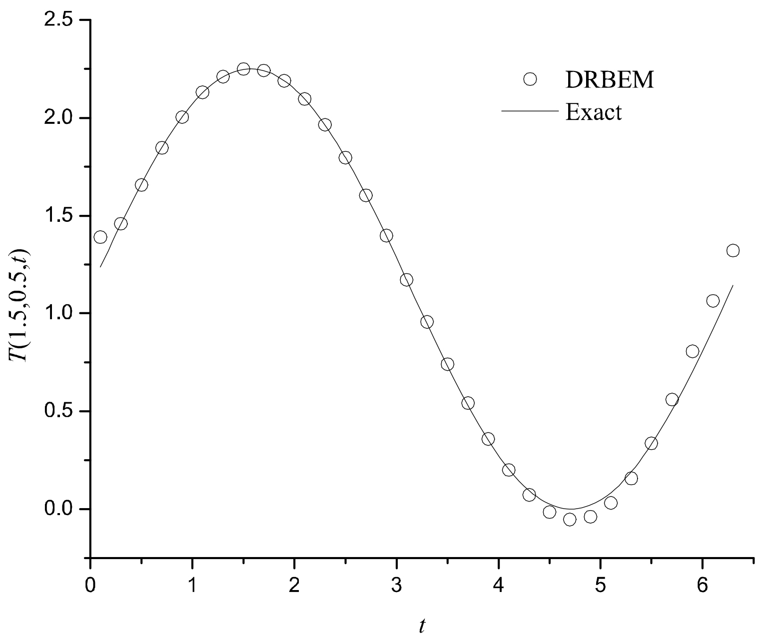

Figure 1. A comparison of the numerical and exact T at $(r, z) = (1.5, 0.5)$ for $0 < t < 6$.

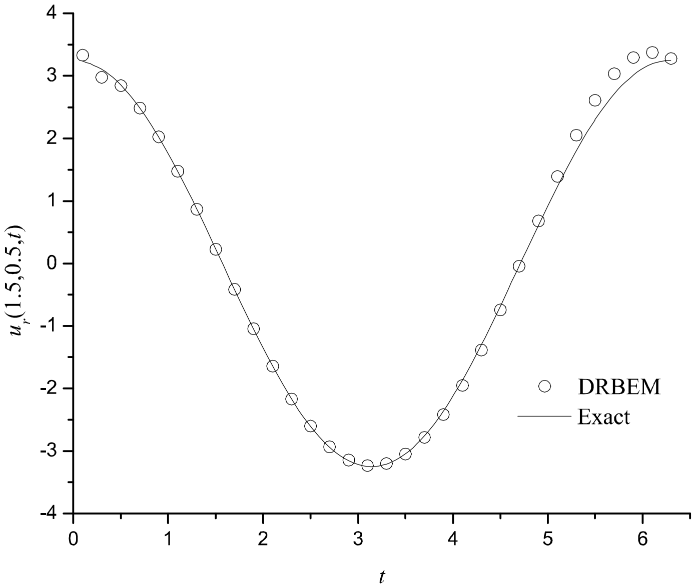

Figure 2. A comparison of the numerical and exact u_r at $(r,z)=(1.5,0.5)$ for $0<t<6$.

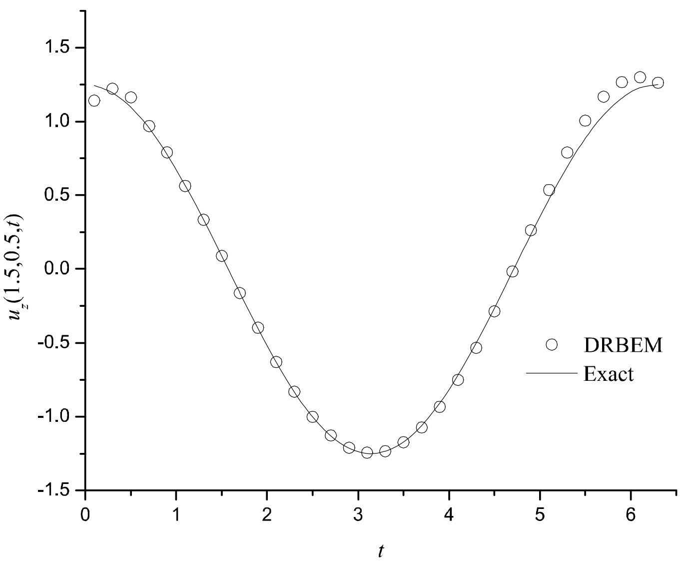

Figure 3. A comparison of the numerical and exact u_z at $(r,z)=(1.5,0.5)$ for $0<t<6$.

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