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Formulation and development of the cell-based smoothed discrete shear gap plate element (CS-FEM-DSG3) using three-node triangles

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Abstract

The paper presents the formulation and recent development of the cell-based smoothed discrete shear gap plate element (CS-FEM-DSG3) using three-node triangles. In the CS-FEM-DSG3, each triangular element will be divided into three sub-triangles, and in each sub-triangle, the original plate element DSG3 is used to compute the strains and to avoid the transverse shear locking. Then the cell-based strain smoothing technique (CS-FEM) is used to smooth the strains on these three sub-triangles. Due to its superior and simple properties, the CS-FEM-DSG3 has been now developed for some different analyses such as: flat shells, stiffened plates, FGM plates, and piezoelectricity composite plates, etc.

Keywords: Reissner-Mindlin plate, smoothed finite element methods (S-FEM), cell-based smoothed finite element method (CS-FEM), cell-based smoothed discrete shear gap method (CS-FEM-DSG3), strain smoothing technique.

Introduction

In the past 50 years, many of plate bending elements based on the Mindlin–Reissner theory and the first-order shear deformation theory (FSDT) have been proposed. Such a large amount of elements can be found in literatures [Reddy (2006)]. In formulations of a Mindlin–Reissner plate element using the FSDT, the deflection *w* and rotations β_x, β_y are independent functions and required at least to be *C*0-continuous. In practical applications, lower-order displacement-based Reissner-Mindlin plate elements are preferred due to their simplicity and efficiency. These elements usually possess high accuracy and fast convergence speed for displacement solutions [Ayad *et al.* (2002)]. In addition, the main difficulty encountered of these elements is the phenomenon of shear locking which induces over-stiffness as the plate becomes progressively thinner.

In order to avoid shear locking, many new numerical techniques and effective modifications have been proposed and tested. Recently, the Discrete-Shear-Gap (DSG) method [Bletzinger *et al.* (2000)] which avoids shear locking was proposed. The DSG method works for elements of different orders and shapes and has several superior properties [Bletzinger *et al.* (2000)]. However, the element stiffness matrix in the DSG still depends on the sequence of node numbers, and hence the solution of DSG is influenced when the sequence of node numbers changes, especially for the coarse and distorted meshes.

In the front of the development of numerical methods, Liu *et al.* have recently integrated the strain smoothing technique [Chen *et al.* 2001] into the point interpolation method (PIM) [Liu *et al.* (2003, 2004a, 2004b)] to create a series of smoothed PIM (S-PIM) [Liu *et al.* (2006a, 2006b, 2013), Zhang *et al.* (2007)], as well as into the FEM to create a series of smoothed FEM (S-FEM) [Liu *et al.* (2010a)] such as the cell/element-based smoothed FEM (CS-FEM) [Liu *et al.* (2007a, 2007b), Dai *et al.* (2007a, 2007b)], the node-based smoothed FEM (NS-FEM) [Liu *et al.* (2009a)], the edge-based smoothed FEM (ES-FEM) [Liu *et al.* (2009b)] and the face-based smoothed FEM (FS-FEM) [Nguyen-Thoi *et al.* (2009a)]. Each of these smoothed FEM has different properties and has been used to produce desired solutions for a wide class of benchmark and practical mechanics problems. Several theoretical aspects of the S-FEM models have been provided in Refs [Liu *et al.* (2007a, 2010b)]. The S-FEM models have also been further investigated and applied to various problems such as plates and shells [Nguyen-Xuan *et al.* (2009a,b), Nguyen-Thoi *et al*. (2013a)], piezoelectricity [Nguyen-Xuan *et al.* (2009c)], visco-elastoplasticity [Nguyen-Thoi *et al.* (2009b)], limit and shakedown analysis for solids [Nguyen-Xuan et al. (2012)], fracture mechanics [Liu *et al.* (2010c)], and some other applications [Nguyen-Thoi *et al*. (2013b,c)], etc.

Among these S-FEM models, the CS-FEM [Liu *et al.* (2007a, 2007b), Dai *et al.* (2007a, 2007b)] shows some interesting properties in the solid mechanics problems. Extending the idea of the CS-FEM to plate structures, Nguyen-Thoi *et al.* (2012) have recently formulated a cell-based smoothed discrete shear gap method (CS-FEM-DSG3). In the CS-FEM-DSG3, each triangular element will be divided into three sub-triangles, and in each sub-triangle, the original plate element DSG3 [Bletzinger *et al.* (2000] is used to compute the strains and to avoid the transverse shear locking. Then the cell-based strain smoothing technique (CS-FEM) is used to smooth the strains on these three sub-triangles. The numerical results showed that the CS-FEM-DSG3 is free of shear locking and achieves the high accuracy compared to the exact solutions and others existing elements.

This paper hence aims to present a brief outline of the CS-FEM-DSG3 and its recent developments in some different analyses such as: flat shells [Nguyen-Thoi *et al.* (2013d)], stiffened plates [Nguyen-Thoi *et al.* (2013e)], FGM plates [Phung-Van *et al.* (2013a)] and piezoelectricity plates [Phung-Van *et al.* (2013b)], etc.

Weakform for the Reissner-Mindlin plate

Consider a plate under bending deformation. The middle surface of plate is chosen as the reference plane that occupies a domain $\Omega \subset R^2$ as shown in Figure 1.

Let *w* be the transverse displacement (deflection), and $\boldsymbol{\beta}^T = \begin{bmatrix} \beta_x & \beta_y \end{bmatrix}$ be the vector of rotations, in which β_x, β_y are the rotations of the middle plane around *y*-axis and *x*–axis, respectively, with the positive directions defined as shown in Figure 1.

The unknown vector of three independent field variables at any point in the problem domain of the Reissner-Mindlin plates can be written as $\mathbf{u}^T = \begin{bmatrix} w & \beta_x & \beta_y \end{bmatrix}$. The curvature of the deflected plate $\boldsymbol{\kappa}$ and the shear strains $\boldsymbol{\gamma}$ are defined, respectively, as

$$\boldsymbol{\kappa} = \mathbf{L}_d \boldsymbol{\beta} \quad ; \quad \boldsymbol{\gamma} = \nabla w + \boldsymbol{\beta} \tag{1}$$

where $\nabla = \begin{bmatrix} \partial / \partial x & \partial / \partial y \end{bmatrix}^T$, and $\mathbf{L}_d$ is a differential operator matrix.

The standard Galerkin weakform of the static equilibrium equations for the Reissner-Mindlin plate can now be written as:

$$\int_{\Omega} \delta \boldsymbol{\kappa}^T \mathbf{D}^b \boldsymbol{\kappa} \, d\Omega + \int_{\Omega} \delta \boldsymbol{\gamma}^T \mathbf{D}^s \boldsymbol{\gamma} \, d\Omega = \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} \, d\Omega \tag{2}$$

where $\mathbf{b}$ is the distributed load applied on the plate. The matrices $\mathbf{D}^b$ and $\mathbf{D}^s$ are the material matrices related to the bending and shear deformation.

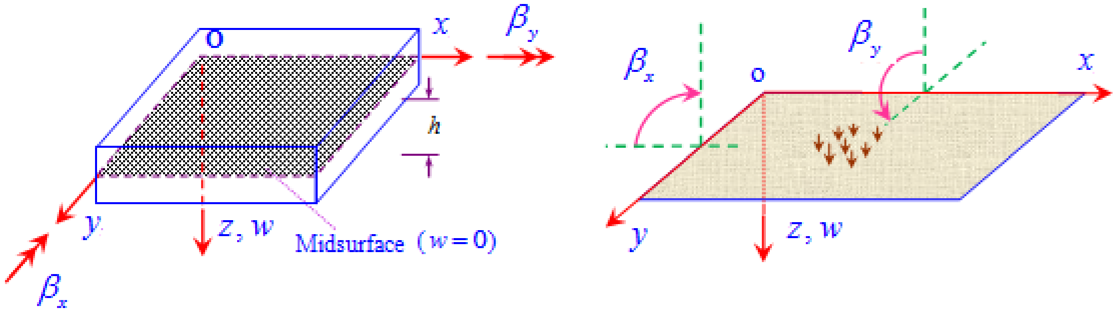

Figure 1. Mindlin plate and positive directions of deflection *w* and rotations β_x, β_y

Formulation of the CS-FEM- DSG3

A brief outline on the formulation of DSG3

Using a mesh of N_e triangular elements such that $\Omega = \bigcup_{e=1}^{N_e} \Omega_e$ and $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, the approximation $\mathbf{u}^h = \begin{bmatrix} w & \beta_x & \beta_y \end{bmatrix}^T$ for a three-node triangular element Ω_e shown in Figure 2 for the Reissner-Mindlin plate can be written, at the element level, as

$$\mathbf{u}_e^h = \sum_{I=1}^{3} \underbrace{\begin{bmatrix} N_I(\mathbf{x}) & 0 & 0 \\ 0 & N_I(\mathbf{x}) & 0 \\ 0 & 0 & N_I(\mathbf{x}) \end{bmatrix}}_{\mathbf{N}_I(\mathbf{x})} \mathbf{d}_{eI} = \sum_{I=1}^{3} \mathbf{N}_I(\mathbf{x}) \mathbf{d}_{eI} \tag{3}$$

where $\mathbf{d}_{eI} = [w_I \;\; \beta_{xI} \;\; \beta_{yI}]^T$ are the nodal degrees of freedom of $\mathbf{u}_e^h$ associated to node *I* and $N_I(\mathbf{x})$ is linear shape functions in a natural coordinate defined by

$$N_1 = 1 - \xi - \eta, \quad N_2 = \xi, \quad N_3 = \eta \tag{4}$$

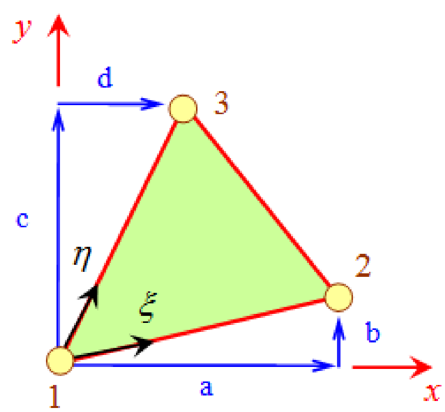

Figure 2. Three-node triangular element and local coordinates in the DSG3.

The bending and shear strains can be then expressed in the matrix forms as:

$$\boldsymbol{\kappa} = \mathbf{B}\mathbf{d}_e, \qquad \boldsymbol{\gamma} = \mathbf{S}\mathbf{d}_e \tag{5}$$

where $\mathbf{d}_e = \begin{bmatrix}\mathbf{d}_{e1} & \mathbf{d}_{e2} & \mathbf{d}_{e3}\end{bmatrix}^T$ is the nodal displacement vector of element, $\mathbf{B}$ and $\mathbf{S}$ contain the derivatives of the shape functions that are constants such as

$$\mathbf{B} = \frac{1}{2A_e}\Bigg[\underbrace{\begin{matrix}0 & b-c & 0\\ 0 & 0 & d-a\\ 0 & d-a & b-c\end{matrix}}_{\mathbf{B}_1}\;\underbrace{\begin{matrix}0 & c & 0\\ 0 & 0 & -d\\ 0 & -d & c\end{matrix}}_{\mathbf{B}_2}\;\underbrace{\begin{matrix}0 & -b & 0\\ 0 & 0 & a\\ 0 & a & -b\end{matrix}}_{\mathbf{B}_3}\Bigg] = \frac{1}{2A_e}\begin{bmatrix}\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3\end{bmatrix} \tag{6}$$

$$\mathbf{S} = \frac{1}{2A_e}\Bigg[\underbrace{\begin{matrix}b-c & A_e & 0\\ d-a & 0 & A_e\end{matrix}}_{\mathbf{S}_1}\;\underbrace{\begin{matrix}c & ac/2 & bc/2\\ -d & -ad/2 & -bd/2\end{matrix}}_{\mathbf{S}_2}\;\underbrace{\begin{matrix}-b & -bd/2 & -bc/2\\ c & ad/2 & ac/2\end{matrix}}_{\mathbf{S}_3}\Bigg] = \frac{1}{2A_e}\begin{bmatrix}\mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3\end{bmatrix} \tag{7}$$

with a, b, c, d are geometric distances as shown in Figure 2 and A_e is the area of the element Ω_e.

Substituting Eqs. (3) and (5) into Eq.(2), the global stiffness matrix now becomes

$$\mathbf{K} = \sum_{e=1}^{N_e}\mathbf{K}_e \tag{8}$$

where $\mathbf{K}_e$ is the element stiffness matrix and is computed by

$$\mathbf{K}_e = \int_{\Omega_e}\mathbf{B}^T\mathbf{D}^b\mathbf{B}\,\mathrm{d}\Omega + \int_{\Omega_e}\mathbf{S}^T\mathbf{D}^s\mathbf{S}\,\mathrm{d}\Omega = \mathbf{B}^T\mathbf{D}^b\mathbf{B}A_e + \mathbf{S}^T\mathbf{D}^s\mathbf{S}A_e \tag{9}$$

Basing on the formulation, it is seen that the element stiffness matrix in the DSG3 depends on the sequence of node numbers of elements, and hence the solution of DSG3 is influenced when the sequence of node numbers of elements changes, especially for the coarse and distorted meshes. The CS-FEM-DSG3 is hence proposed to overcome this drawback and also to improve the accuracy as well as the stability of the DSG3.

Formulation of CS-FEM-DSG3

In the CS-FEM-DSG3 [Nguyen-Thoi *et al.* (2012)], the domain discretization is the same as that of the DSG3 [Bletzinger *et al.* (2000)] using N_n nodes and N_e triangular elements. However in the formulation of the CS-FEM-DSG3, each triangular element is divided into three sub-triangles by connecting the central point O of the element to three field nodes as shown in Figure 3. Using the DSG3 [Bletzinger *et al.* (2000)] formulation for each sub-triangle, the bending and shear strains in 3 sub-triangles are then obtained, respectively, by

$$\boldsymbol{\kappa}_e^{\Delta_j} = \mathbf{B}^{\Delta_j}\mathbf{d}_e \quad , \; j = 1,2,3 \tag{10}$$

$$\boldsymbol{\gamma}_e^{\Delta_j} = \mathbf{S}^{\Delta_j}\mathbf{d}_e \quad , \; j = 1,2,3 \tag{11}$$

where $\mathbf{d}^e$ is the vector containing the nodal degrees of freedom of the element; $\mathbf{B}^{\Delta_j}$, $\mathbf{S}^{\Delta_j}$, j = 1,2,3, are bending and shearing gradient matrices by the DSG3 [Bletzinger *et al.* (2000)] of j^{th} sub-triangle, respectively.

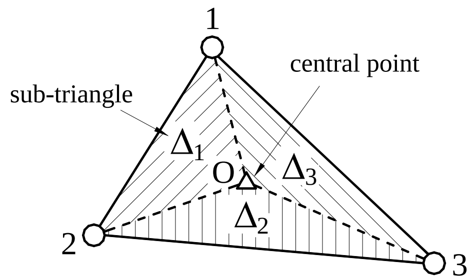

Figure 3. Three sub-triangles (Δ_1, Δ_2 and Δ_3) created from the triangle 1-2-3 in the CS-FEM-DSG3 by connecting the central point O with three field nodes 1, 2 and 3.

Now, applying the cell-based strain smoothing operation in the CS-FEM [Liu *et al.* (2010a)], the bending and shear strains $\boldsymbol{\kappa}_e^{\Delta_j}$, $\boldsymbol{\gamma}_e^{\Delta_j}$, $j = 1, 2, 3$ are, respectively, used to create element smoothed strains $\tilde{\boldsymbol{\kappa}}_e$ and $\tilde{\gamma}^e$ on the triangular element Ω_e, such as:

$$\tilde{\boldsymbol{\kappa}}_e = \tilde{\mathbf{B}}\mathbf{d}_e \quad ; \quad \tilde{\boldsymbol{\gamma}}_e = \tilde{\mathbf{S}}\mathbf{d}_e \tag{12}$$

where $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{S}}$ are the smoothed strain gradient matrices, respectively, given by

$$\tilde{\mathbf{B}} = \frac{1}{A_e}\sum_{j=1}^{3} A_{\Delta_j}\mathbf{B}^{\Delta_j} \quad ; \quad \tilde{\mathbf{S}} = \frac{1}{A_e}\sum_{j=1}^{3} A_{\Delta_j}\mathbf{S}^{\Delta_j} \tag{13}$$

Therefore the global stiffness matrix of the CS-FEM-DSG3 is computed by

$$\tilde{\mathbf{K}} = \int_\Omega \tilde{\mathbf{B}}^T\mathbf{D}^b\tilde{\mathbf{B}}\mathrm{d}\Omega + \int_\Omega \tilde{\mathbf{S}}^T\mathbf{D}^s\tilde{\mathbf{S}}\mathrm{d}\Omega = \tilde{\mathbf{B}}^T\mathbf{D}^b\tilde{\mathbf{B}}A_e + \tilde{\mathbf{S}}^T\mathbf{D}^s\tilde{\mathbf{S}}A_e \tag{14}$$

Advantages of CS-FEM-DSG3

Through the formulation of CS-FEM-DSG3 [Nguyen-Thoi *et al.* (2012)], it is seen that the method is simple and only based on three-node triangular elements without adding any additional DOFs. The CS-FEM-DSG3 is free of shear locking and pass the patch test. The method can be seen as an effective tool for analyses of Mindlin plates. Through the numerical examples of CS-FEM-DSG3 [Nguyen-Thoi *et al.* (2012)], the method shows four superior properties such as: (1) be a strong competitor to many existing three-node triangular plate elements in the static analysis; (2) can give high accurate solutions for problems with skew geometries in the static analysis; (3) can give high accurate solutions in free vibration analysis; (4) can provide accurately the values of high frequencies of plates by using only coarse meshes.

Extension of the CS-FEM-DSG3 to some others applications

Due to its superior and simple properties, the CS-FEM-DSG3 has been extended quickly to some different analyses.

First, by adding three degrees of freedom of the membrane and rotation displacements, together using the coordinate transformation matrix, the CS-FEM-DSG3 is easy to extend to the flat shell element [Nguyen-Thoi *et al.* (2013d)]. This extention hence highlights the advantage of the method which uses only tringular elements, because the geometry of shell structures is often much more complicated than that of the plate structures.

Next, by combining with a membrane element and stiffened by a thick beam element Timoshenko, the CS-FEM-DSG3 is extended to the stiffened plates [Nguyen-Thoi *et al.* (2013e)]. In this element, the eccentricity between the plate and the beam is included in the formulation of the beam. The compatibility of deflection and rotations of stiffeners and plate is assumed at the contact positions.

Next by using 7 degrees of freedom, the CS-FEM-DSG3 is extended to the C^0-type high-order shear deformation plate theory for the static and free vibration analyses of functionally graded plates (FGPs) [Phung-Van *et al.* (2013a)]. In the FGPs, the material properties are assumed to vary through the thickness by a simple power rule of the volume fractions of the constituents. In the static analysis, both thermal and mechanical loads are considered and a two-step procedure is performed including a step of analyzing the temperature field along the thickness of the plate and a step of analyzing the behavior of the plate subjected to both thermal and mechanical loads.

And recently, by combining the degrees of membrane displacement and electric potential, the CS-FEM-DSG3 is further extended for the static, free vibration analyses and dynamic control of composite plates integrated with piezoelectric sensors and actuators [Phung-Van *et al.* (2013b)]. In the piezoelectric composite plates, the electric potential is assumed to be a linear function through the thickness of each piezoelectric sublayer. A displacement and velocity feedback control algorithm is used for the active control of the static deflection and the dynamic response of plates through the closed loop control with bonded or embedded distributed piezoelectric sensors and actuators.

Conclusions

The paper presents a brief outline and recent developments of the CS-FEM-DSG3 using three-node triangles. In the original plate element, each triangular element will be divided into three sub-triangles, and in each sub-triangle, the original plate element DSG3 is used to compute the strains and to avoid the transverse shear locking. Then the cell-based strain smoothing technique (CS-FEM) is used to smooth the strains on these three sub-triangles. Through the formulation and numerical examples, it is seen that the CS-FEM-DSG3 is an effective tool for analyses of Mindlin plates. And due to its superior and simple properties, the CS-FEM-DSG3 has been extended quickly to some different analyses such as flat shells, stiffened plates, FGM plates, and piezoelectricity composite plates, etc.

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