Direct determination of critical load combinations for elastoplastic structures subject to multiple load cases

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Key Words: *Structural design, Multiple load cases, Elastoplasticity, Integer program*

Abstract

The paper presents a mathematical programming based approach for the safety assessment of nonlinear structures that can be subject to a number of predefined load combinations. The objective is to determine in a single step the critical load combination, for which the chosen maximum (or minimum) response (e.g. stress or displacement) occurs. Assuming elastoplastic material properties, the governing formulation takes the form of challenging nonconvex and nonsmooth optimization problem, known as a 0-1 mathematical program with equilibrium constraints or 0-1 MPEC. The numerical algorithm proposed to solve the 0-1 MPEC is a regularization technique that involves iteratively processing a series of reformulated mixed integer nonlinear programming problems (MINLP) using a penalty function. Optimal solutions to each MINLP subproblem are obtained by the proposed novel space search formulation or FSS scheme.

Keywords: Structural design, Multiple load cases, Elastoplasticity, Integer program

Introduction

The safety assessment of structures under multiple load cases provides a well-accepted numerical approximation for the critical structural responses, such as the extreme maximum and minimum bound values to some specified variables of interest, e.g. member forces, nodal displacement, etc. Various papers in this area (see e.g. Mullen and Muhanna, 1999; Suarjana and Law, 1994) have established reliable theoretical and numerical treatments, albeit solely for elastic material related problems. It is useful, and often mandatory, to incorporate the influence of material nonlinear properties for the realistic assessment of practical engineering mechanics applications.

The present study proposes a pair of novel mathematical programming based approaches to directly identify the maximum bound value in one case and the minimum bound value in the other case to some selected set of response variables of an elastoplastic structure subjected to various pattern load cases. The governing problem takes the "nonstandard" and difficult form known in the nonconvex and/or nonsmooth optimization theory as a 0-1 mathematical program with equilibrium constraints or 0-1 MPEC (Kocis and Grossmann, 1989; Luo et al., 1996). The specific equilibrium constraints are characterized by complementarity conditions (representing plastic behavior). We propose a penalty regularization technique to transform the challenging original 0-1 MPEC to a standard mixed integer nonlinear programming or MINLP problem, and then successively solve a series of MINLP subproblems to iteratively enforce the complementarity. A novel formulation space search or FSS algorithm (López and Beasley, 2013) is adopted to capture the best (optimal) solution for each of the MINLP subproblems. We illustrate the application and robustness of the proposed method through one of the many examples we have solved.

Critical Load Combinations as a 0-1 MPEC

Our work adopts a standard "line" finite element framework, where the structure is suitably discretized into *n* elements, *m* basic stresses/strains, *d* degrees of freedom and *y* yield functions. The nonlinear material properties are accommodated through the classical plastic hinge concept. Plasticity is confined solely at fixed critical zones, namely the two ends for each generic element, whilst the rest of the member between these ends remains elastic. A computationally advantageous piecewise linear plastic concept (Maier, 1970) is adopted to provide a close approximation to the actual nonlinear yield surfaces, as a number of linear hyperplanes.

The generic formulations describing the path-independent or holonomic elastoplastic response of the structure under a single load case are written as follows (see e.g. Maier, 1970; Tangaramvong and Tin-Loi, 2007, 2008):

$$\mathbf{C}^{\mathrm{T}}\mathbf{Q} = \mathbf{F}, \tag{1}$$

$$\mathbf{Cu} = \mathbf{e} + \mathbf{p}, \tag{2}$$

$$\mathbf{Q} = \mathbf{Ee}, \tag{3}$$

$$\mathbf{p} = \mathbf{N}\boldsymbol{\lambda}, \tag{4}$$

$$\mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{r} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda} = 0. \tag{5}$$

More explicitly, equilibrium between basic stresses $\mathbf{Q} \in \Re^{m}$ and externally applied forces $\mathbf{F} \in \Re^{d}$ is given in Eq. (1) through a (constant) compatibility matrix $\mathbf{C} \in \Re^{m\times d}$. The linear compatibility relation between nodal displacements $\mathbf{u} \in \Re^{d}$ and basic strains, written as a summation of elastic $\mathbf{e} \in \Re^{m}$ and plastic $\mathbf{p} \in \Re^{m}$ strains, is described in Eq. (2). The elastic constitutive behaviors are expressed in Eq. (3) using a positive definite stiffness matrix $\mathbf{E} \in \Re^{m\times m}$.

The associative flow rule in Eq. (4) defines the plastic strains $\mathbf{p}$ as functions of plastic multipliers $\boldsymbol{\lambda} \in \Re^{y}$ through a constant normality matrix $\mathbf{N} \in \Re^{m\times y}$, which collects unit normal directions to all piecewise linear yield hyperplanes. Finally, the complementarity condition (viz. $\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda} = 0$) in Eq. (5) between the two sign-constrained variables, namely the yield functions $\mathbf{w} \geq \mathbf{0} \in \Re^{y}$ and the plastic multipliers $\boldsymbol{\lambda} \geq \mathbf{0}$ describes the inherent holonomic structural behavior that permits reversal of plastic strains at the potential plastic hinges, where $\mathbf{r} \in \Re^{y}$ is a vector of yield limits.

Simply collecting and manipulating the governing holonomic Eqs (1) to (5) provides the following state problem in mixed static-kinematic variables ($\mathbf{Q},\mathbf{u},\boldsymbol{\lambda}$):

$$\begin{aligned}&\mathbf{C}^{\mathrm{T}}\mathbf{Q} - \mathbf{F} = \mathbf{0},\\ &\mathbf{Q} - \mathbf{ECu} + \mathbf{EN}\boldsymbol{\lambda} = \mathbf{0},\\ &\mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{r} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda} = 0.\end{aligned} \tag{6}$$

The state problem given in Eq. (6) is a mixed complementarity problem or MCP (Dirkse and Ferris, 1995a). For any predefined force vector $\mathbf{F}$, the key response variables ($\mathbf{Q},\mathbf{u},\boldsymbol{\lambda}$) can be obtained by solving the MCP (6) using, for instance, the state-of-the-art complementarity solver, namely PATH (Dirkse and Ferris, 1995b), that is available from within the general algebraic modeling system or GAMS (Brooke et al., 1998), that we adopted for this work.

When pattern load cases, as for practical design instances, are considered, it is essential that the critical load combination leading to the worst value to some response variables of interest is determined. An attempt to achieve this typically involves exhaustive trial-and-error and often requires excessive computing and designer-time resources, especially when a large number of load cases are concerned. To circumvent this, the present method develops a robust optimization technique that determines, within a single step, the critical load case associated with the maximum (or minimum) response variable of the structure under all possible load combinations.

We first replace the known force vector **F** in Eq. (6) with the set of all s possible predefined load cases $\mathbf{f} \in \Re^{d \times s}$ and additional unknown binary (0 or 1) variables $\boldsymbol{\alpha} \in \Re^{s}$:

$$\mathbf{F} = \mathbf{f}\boldsymbol{\alpha}. \tag{7}$$

The variables $\boldsymbol{\alpha}$ play the crucial role in automatically selecting which particular load case $\mathbf{f}^i \in \Re^d$ for $i = 1$ to s is retained (viz. $\alpha^i = 1$) or eliminated ($\alpha^i = 0$) from the structural system. A direct determination is then enabled by forming an optimization problem, in variables $(\boldsymbol{\alpha},\mathbf{Q},\mathbf{u},\boldsymbol{\lambda})$, that maximizes (or minimizes) an objective function representing the specific response variable Y (i.e. some basic stress Q^i, nodal displacement u^i, etc.), subject to the constraints describing the holonomic elastoplastic relations in Eq. (6) and the multiple load cases in Eq. (7):

$$\begin{aligned} &\mathit{max}\,(\text{or } \mathit{min}) \;\; Y \\ &\mathit{subject\ to} \quad \mathbf{C}^{\mathrm{T}}\mathbf{Q} - \boldsymbol{\alpha}\mathbf{f} = \mathbf{0}, \\ &\qquad\qquad\quad \mathbf{Q} - \mathbf{ECu} + \mathbf{EN}\boldsymbol{\lambda} = \mathbf{0}, \\ &\qquad\qquad\quad \mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{r} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda} = 0. \end{aligned} \tag{8}$$

The problem in Eq. (8) belongs to the challenging class of "nonstandard" optimization programs, known as a 0-1 MPEC (Kocis and Grossmann, 1989; Luo et al., 1996). In addition to the difficulties associated with the presence of complementarity constraints making the problem severely nonconvex and/or nonsmooth, the binary variables $\boldsymbol{\alpha}$ impart computationally nasty disjunctive and combinatorial conditions to the 0-1 MPEC (8). To date, there are no known algorithms that can guarantee the (global) optimality to the solutions of MPECs, let alone of 0-1 MPECs. The best method is often dependent on the nature of the specific problem.

Penalty-FSS Algorithm

In this section, we propose a combined penalty regularization (Tangaramvong and Tin-Loi, 2007, 2008) and FSS (López and Beasley, 2013) technique as a scheme to obtain the optimal solutions of the 0-1 MPEC (8). The 0-1 MPEC (8) is first reformulated as a standard MINLP problem by replacing the complementarity condition with the penalized term (viz. $\mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}$ or $-\mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}$) added directly to the objective function, where μ denotes a (positive scalar) penalty parameter. We attempt to enforce the complementarity condition by iteratively increasing the parameter μ. Thus, the penalty algorithm we use processes a series of MINLP subproblems, each represented by

$$\begin{aligned} &\mathit{max}\,(\text{or } \mathit{min}) \;\; Y - \mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda} \;\; (\text{or } Y + \mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}) \\ &\mathit{subject\ to} \quad \mathbf{C}^{\mathrm{T}}\mathbf{Q} - \boldsymbol{\alpha}\mathbf{f} = \mathbf{0}, \\ &\qquad\qquad\quad \mathbf{Q} - \mathbf{ECu} + \mathbf{EN}\boldsymbol{\lambda} = \mathbf{0}, \\ &\qquad\qquad\quad \mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{r} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0}, \end{aligned} \tag{9}$$

with increasing μ (e.g. $\mu = 10\mu$) until the preset tolerance on the original complementarity condition (e.g. $\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda} \leq 10^{-6}$) is satisfied.

The success of the penalty method relies on the ability to capture the optimal solutions for each of the MINLP subproblems (9). Unfortunately, directly processing such a problem entails severe combinatorial difficulties and is likely to fail in generating optimal results. A better numerical (albeit heuristic) method, namely FSS (López and Beasley, 2013), is proposed to circumvent these challenges, by simply adding the following additional nonlinear constraint to increase the "tightness" of the original MINLP (9):

$$\boldsymbol{\alpha}^{\mathrm{T}}(\mathbf{1}-\boldsymbol{\alpha}) \leq \gamma, \tag{10}$$

where γ is a (positive) relaxation parameter to FSS and $\mathbf{1}$ a unit vector. Therefore, the original MINLP subproblem (9) can be rewritten as:

$$\begin{aligned} &\mathit{max}\,(\text{or } \mathit{min}) && Y-\mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}\ \ (\text{or } Y+\mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}) \\ &\textit{subject to} && \mathbf{C}^{\mathrm{T}}\mathbf{Q}-\boldsymbol{\alpha}\mathbf{f}=\mathbf{0}, \\ & && \mathbf{Q}-\mathbf{ECu}+\mathbf{EN}\boldsymbol{\lambda}=\mathbf{0}, \\ & && \mathbf{w}=-\mathbf{N}^{\mathrm{T}}\mathbf{Q}+\mathbf{r}\geq\mathbf{0},\ \boldsymbol{\lambda}\geq\mathbf{0}, \\ & && \boldsymbol{\alpha}^{\mathrm{T}}(\mathbf{1}-\boldsymbol{\alpha})\leq\gamma. \end{aligned} \tag{11}$$

The algorithm attempts to explore, at inner-level iterations, various different sets of MINLP solutions by initially setting γ to some suitable value and subsequently decreasing γ to tighten the binary constraints. After obtaining the multiple sets of MINLP solutions, the best value is selected as the "optimum" to the original MINLP (9).

The proposed penalty-FSS algorithm can be summarized in the pseudocode as follows:

Step (a) – Initialization
- Set: initial γ, μ, maximum penalty iterations (*maxplt*), maximum FSS iterations (*maxfss*), $\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}=100$, $Y_{best}=0$ for maximization (or $Y_{best}=1000$ for minimization), *rpt* = 0

Step (b) – Penalization
- For j = 1 to *maxplt*
    - if $\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}\leq 10^{-6}$, terminate, end
    - go to FSS Step (c) to obtain the solutions of MINLP (9)
    - increase $\mu = 10\mu$

  end

Step (c) – FSS
- For k = 1 to *maxfss*
    - if *rpt* > 3 or $\gamma \leq 10^{-5}$, terminate, end
    - solve MINLP (11)
    - if $Y = Y_{best}$
        - count *rpt* = *rpt* + 1
    - elseif $Y > Y_{best}$ for maximization (or $Y < Y_{\mathrm{best}}$ for minimization)
        - update: $Y_{best} = Y$ and variables ($\boldsymbol{\alpha}_{best}$,$\mathbf{Q}_{best}$,$\mathbf{u}_{best}$,$\boldsymbol{\lambda}_{best}$)
        - reset *rpt* = 0
    - end
    - decrease $\gamma = 0.1\gamma$

  end
- Update: $Y = Y_{best}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{best}$, and so on. Return to Step (b)

In Step (a), we initialize the penalty parameter to $\mu = 1$, and the FSS relaxation parameter γ by processing the following relaxed (continuous but bounded) system to the original MINLP problem (9) in noninteger variables $(\boldsymbol{\alpha},\mathbf{Q},\mathbf{u},\boldsymbol{\lambda})$:

$$\begin{aligned} &\mathit{max}\ (\text{or } \mathit{min})\ \ Y - \mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}\ \ (\text{or } Y + \mu\mathbf{w}^{\mathrm{T}}\boldsymbol{\lambda}) \\ &\mathit{subject\ to}\quad \mathbf{C}^{\mathrm{T}}\mathbf{Q} - \boldsymbol{\alpha}\mathbf{f} = \mathbf{0}, \\ &\qquad\qquad\quad\ \mathbf{Q} - \mathbf{ECu} + \mathbf{EN}\boldsymbol{\lambda} = \mathbf{0}, \\ &\qquad\qquad\quad\ \mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{r} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0}, \\ &\qquad\qquad\quad\ \mathbf{0} \leq \boldsymbol{\alpha} \leq \mathbf{1}, \end{aligned} \tag{12}$$

and hence from the solutions obtained in Eq. (12)

$$\gamma = \boldsymbol{\alpha}^{\mathrm{T}}(\mathbf{1} - \boldsymbol{\alpha}). \tag{13}$$

It is clear that the optimization given in Eq. (12) is a standard nonlinear programming or NLP problem that can be easily solved by any available NLP code, such as GAMS/CONOPT (Drud, 1994). Furthermore, the solutions obtained from the relaxed NLP (12) provide a good approximation to the initial variables of MINLP (11).

In Step (b), each MINLP (9) subproblem is processed using the FSS scheme in Step (c), where a series of increasingly "tighter" MINLPs (11) is solved by systematically reducing the value of γ. Thus, different sets of solutions to MINLP (11) are generated, and the best optima are updated. The FSS Step (c) terminates when either the MINLP (11) solve finds more than three consecutive identical solutions or the FSS parameter γ is sufficiently small, namely $\gamma \leq 10^{-5}$.

We process the MINLP solve using the GAMS/DICOPT solver (Kocis and Grossmann, 1989). The proposed penalty-FSS algorithm is coded within a MATLAB framework that is linked directly to the GAMS environment through a MATLAB-GAMS interface software (Ferris, 1998).

Illustrative Example

The three-span continuous beam in Fig. 1 with three identical vertical point loads of 80 kN, each independently applied at midspan, is considered. This structure was previously investigated by Mullen and Muhanna (1999) to illustrate applications of the fuzzy finite element method to obtain the maximum and minimum bounds to the bending moments under multiple load combinations, where only elastic material properties were assumed.

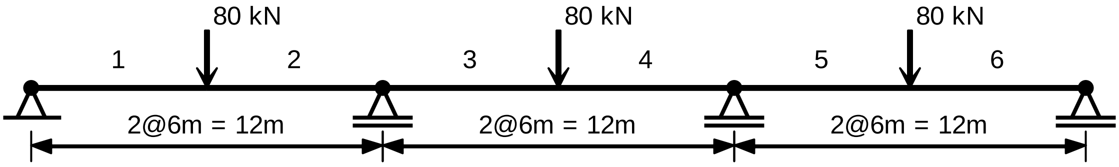

Figure 1. Three-span continuous beams under multiple load cases.

In the present study, we further incorporate the influences of elastic perfectly plastic material (steel) properties: modulus of elasticity of $E = 2 \times 10^8$ kNm^{-2}; cross sectional area of $A = 6000 \times 10^{-6}$ m^2; second moment of area of $I = 30 \times 10^{-6}$ m^4; and flexural force plastic capacity of $Q_{2\mathrm{u}} = 175$ kNm.

As is practical for the beam structure, a pure bending yield model is adopted for each of the potential plastic hinges, namely at member ends.

The beam discretization consists of 6 elements, 18 basic stresses/strains, 16 degrees of freedom and 24 yield functions. The pattern loads generated by the three point loads (Fig. 1) lead to 8 possible combinations (e.g. one point load at each span, two point loads at two adjacent spans, etc.).

Table 1. Critical bending moments and associated load patterns by penalty-FSS method.

	Element 1		Element 2		Element 3	
	Q_2^1	Q_3^1	Q_2^2	Q_3^2	Q_2^3	Q_3^3
Maximum Y	–	175 {A}	36 {B}	32.5 {C}	168 {D}	168 {B}
Minimum Y	–	–36 {B}	–175 {A}	–168 {D}	–32.5 {C}	–130 {E}

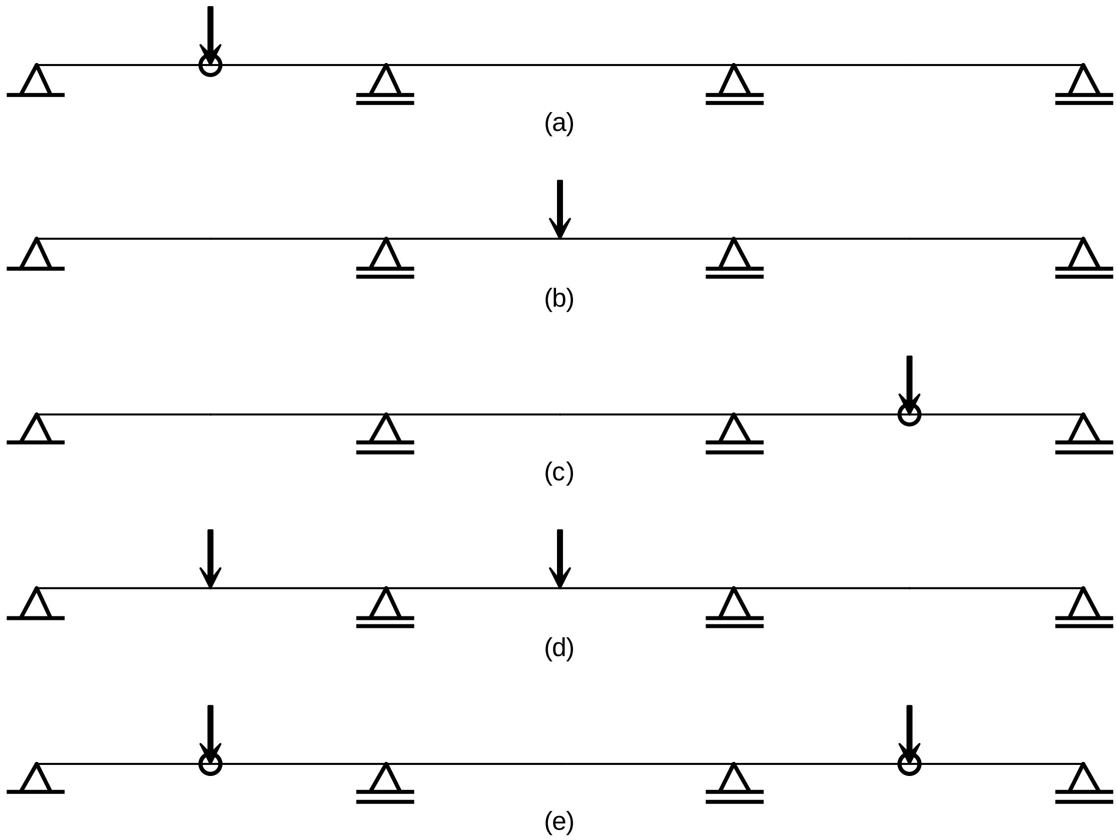

Figure 2. Critical load combinations (a) pattern A, (b) pattern B, (c) pattern C, (d) pattern D and (e) pattern E, where o denotes plastic hinge.

The proposed penalty-FSS algorithm successfully computed the critical (viz. the most maximum and the least minimum) bounds to flexural forces (unit kNm) at the two start Q_2^i and end Q_3^i nodes for each of the members 1 to 6. In view of symmetry of the structural system, the obtained results of a half beam, namely members 1 to 3, are summarized in Table 1, where positive and negative signs define clockwise and anticlockwise flexural directions, respectively. The critical load patterns (also indicated in brackets in Table 1) corresponding to each of the 0-1 MPEC solutions are displayed in Fig. 2. The CPU times are not reported as each of the 0-1 MPEC (8) solves only took few seconds to furnish the results for the modest 2.7-GHz Pentium personal computer with 3-GB RAM running WinXP.

The accuracy of these optimal solutions is validated through a comparison with the results found from the complete (exhaustive) MCP (6) solves involving 8 possible load combinations.

Conclusions

A mathematical programming based penalty-FSS method has been proposed for the direct determination of the critical bound to some response variables of structures under multiple load cases. The present study incorporates the influences of material (elastic perfectly plastic) nonlinearity, as is necessary for the realistic safety assessment of the structures. The key feature of the proposed scheme is to compute, within a single step, the most maximum in one case and the least minimum in the other case response values of interest for the structure subjected to known design multiple load combinations. The specific pattern load corresponding to each of the critical bounds is obtained as a by-product.

The governing problem takes the form of a challenging optimization problem, known in the nonconvex and/or nonsmooth optimization literature as a 0-1 MPEC. To circumvent the difficulties associated with nonconvexity and disjunctiveness, a penalty-FSS algorithm has been proposed. Such a scheme enforces the complementarity condition by iteratively processing a series of penalized MINLP subproblems with successively increasing penalty parameter. The solutions of are searched at an inner-level enumeration using the FSS technique by introducing an additional "tightness" constraint to the MINLP subproblem. By suitably decreasing the FSS relaxation parameter, various different sets of optimal solutions to the original MINLP subproblem can be identified, and the algorithm then selects the best optimum.

A number of numerical examples, a simple one of which has been provided herein, indicate the robustness and efficiency of the proposed penalty-FSS method. The scheme can accurately capture the desired optimal bound solutions, as validated by computationally expensive exhaustive MCP solves representing all possible load cases.

A straightforward extension to the present scheme is to incorporate various other nonlinear behaviors found in practical engineering mechanics applications, such as nonlinear geometry and physically instabilizing softening materials.

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